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DYNAMICS OF NONLINEAR DIFFERENCE EQUATIONS

Prepared by
Areej Awawdeh

Supervised by
Dr.Marwan Aloqeili

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Prepared by

AREEJ AWAWDEH

Supervised by

Dr. MARWAN ALOQEILI

Birzeit University

Palestine

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled

DYNAMICS OF NONLINEAR DIFFERENCE EQUATIONS

By

AREEJ AWAWDEH

in partial fulfillment of the requirements for the degree of Master.
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Committee Members		Signature
1. Dr. Marwan Aloqeili	Head of Committee	....
2. Prof. Mohammad Saleh	Internal Examiner	....
3. Dr. Raimon Gedeon	Internal Examiner	....

الإهداء

إلى من وهب حياته درعاً ليوصلني إلى برّ الأمان

إلى من رحلت روحه في عتمة الأيام . . . .

أبي الغالي

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DECLARATION

I certify that this thesis, submitted for the degree of Master of Science to the Department of Mathematics in Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis has not been submitted for a higher degree to any other university or institution.

Areej Awawdeh Signature.............

July 9, 2014

Abstract

This research aims mainly to study properties of three different difference equations. The first equation is

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \;\; n = 0, 1, ...,$$

with initial conditions $x_{-1} \geq 0$, $x_0 > 0$, and where $\{p_n\}$ is a positive bounded sequence,. The second equation is

$$x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}, \;\; n = 0, 1, ...,$$

where A_n is a positive bounded sequence, the initial conditions $x_{-1} \geq 0$, $x_0 > 0$, and p, $q \in (0, \infty)$. And the third equation is

$$x_{n+1} = p_n + \frac{x_n}{x_{n-1}}, \;\; n = 0, 1, ...,$$

where $x_{-1} > 0$, $x_0 \geq 0$, and p_n is a positive bounded sequence. For each equation we studied periodicity, stability, attractivity and boundedness.

الــمـلـخص

الهدف من هذا البحث دراسة سلوك المعادلات التفاضلية المنفصلة التالية

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \qquad n = 0,1, \dots,$$

حيث أن $x_{-1} \geq 0, x_0 > 0$ و p_n عبارة عن متتالية موجبة محدودة.

والمعادلة

$$x_{n+1} = A_n + \frac{x_{n-1}^p}{{x_n}^q}, \qquad n = 0,1, \dots,$$

حيث أن $x_{-1} \geq 0, x_0 > 0$ و A_n متتالية موجبة محدودة و $p, q \in (0, \infty)$.

و المعادلة

$$x_{n+1} = p_n + \frac{x_n}{x_{n-1}}, \qquad n = 0,1, \dots,$$

حيث أن $x_{-1} > 0, x_0 \geq 0$ و p_n عبارة عن متتالية موجبة محدودة.

Contents

Chapter 1

Introduction

1.1 Difference Equations

Difference equation is an equation that defines a relation recursively, in other words, each term of the sequence is defined as a function of the previous terms of the sequence. The difference equation of order k is of the form

$$x_n = f(x_{n-1}, x_{n-2}, ..., x_{n-k}), \quad n = 0, 1, 2, ..., \tag{1.1}$$

Starting from a point x_0 for the equation $x_{n+1} = f(x_n)$, you will get the following sequence

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), ...$$

This sequence can be written as

$$x_0, f(x_0), f^2(x_0), f^3(x_0), ...$$

$f(x_0)$ is called the first iterate of x_0 under the function f, $f^2(x_0)$ is the second iterate under f, $f^3(x_0)$ is the third iterate under f. The set of all iterates $\{f^n(x_0) : n \geq 0\}$ where $f^0(x_0) = x_0$ is called the positive orbit of x_0, the orbit will be denoted by $O(x_0)$.

Difference equations can be classified into different categories according to one or more of the following properties:

(1) **Linear difference equations**: an equation is said to be linear if the function f in Eq.(1.1) is a linear function.

(2) **Nonlinear difference equations**: an equation is said to be nonlinear if the function in Eq.(1.1) is a nonlinear function.

(3) **Linear homogeneous difference equations**: a kth-order linear homogeneous difference equation is an equation of the form

$$y_{n+k} + p_1(n)y_{n+k-1} + ... + p_k(n)y_n = 0,$$

where $p_k(n) \neq 0$ for all $n \geq n_0$.

(4) **Linear nonhomogeneous difference equations**: a kth-order linear nonhomogeneous difference equation is an equation of the form

$$y_{n+k} + p_1(n)y_{n+k-1} + ... + p_k(n)y_n = g(n),$$

where $p_k(n) \neq 0$ for all $n \geq n_0$.
The sequence $g(n)$ is called the forcing term.

(5) **Autonomous difference equations**: a kth-order difference equation is said to be autonomous if it is time-invariant, in other words

$$x_n = f(x_{n-1}, x_{n-2}, ..., x_{n-k}).$$

(6) **Nonautonomous difference equations**[8]: a kth-order difference equation is said to be nonautonomous if the function f is replaced by a new function g of two variables, $g : \mathbf{Z}^+ \times \mathbf{R} \to \mathbf{R}$, this can be denoted as

$$x_n = g(n, x_{n-1}, x_{n-2}, ..., x_{n-k}).$$

In this case the equation is time-variant.

(7) **Linear difference equation with constant coefficients**: a kth-order linear difference equation with constant coefficients is an equation of the form

$$x_{n+k} + p_1 x_{n+k-1} + p_2 x_{n+k-2} + ... + p_k x_n = g(n),$$

where p_i's are constants and $p_k \neq 0$ for all $n \geq n_0$.

(8) **Linear difference equation with nonconstant coefficients**: a kth-order linear difference equation with nonconstant coefficients is an equation of the form

$$x_{n+k} + p_1(n)x_{n+k-1} + p_2(n)x_{n+k-2} + ... + p_k(n)x_n = g(n),$$

where $p_k(n) \neq 0$ for all $n \geq n_0$.

1.2 Sequences

A *sequence* x_n of real numbers is a function defined on the set of natural numbers whose range is contained in the set of real numbers.
This can be abbreviated as

$$f : \mathbf{N} \rightarrow \mathbf{R}.$$

Definition 1.2.1 *[5]**The limit of a sequence***
We say that a number x is a limit of the sequence x_n if for each $\epsilon > 0$, there exists a natural number K such that for all $n \geq K$ we have $|x_n - x| < \epsilon$. In symbols

$$(\epsilon > 0)(\exists K)(n \geq K)(|x_n - x| < \epsilon).$$

Definition 1.2.2 *[5]**Bounded sequence***
A sequence x_n of real numbers is said to be bounded if there exists a positive real number M such that $|x_n| \leq M$ for all natural numbers n.

Definition 1.2.3 *[18] If x_n is a sequence, we define the* $\limsup x_n$ *as*

$$\limsup x_n = \inf_n \sup_{k \geq n} x_k.$$

The $\liminf x_n$ *is defined as*

$$\liminf x_n = \sup_n \inf_{k \geq n} x_k.$$

Definition 1.2.4 *Let x_n, $\bar{x}_n$ be two sequences, we say that the sequence x_n converges to the sequence $\bar{x}_n$, in symbols*

$$x_n \to \bar{x}_n$$

if

$$\lim_{n\to\infty} \frac{x_n}{\bar{x}_n} = 1.$$

The sequence $\{x_n\}$ is said to be *m-periodic* if $x_{n+m} = x_n$.

1.3 Behavior of Solutions of Difference Equations

The difference equation of order $k+1$ is of the form

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, 2, ..., \tag{1.2}$$

A point $\bar{x}$ is said to be a *fixed point* of the difference Eq. (1.2) if

$$f(\bar{x}, \bar{x}, ..., \bar{x}) = \bar{x}.$$

Definition 1.3.1 *A point $\bar{x}$ in the domain of f is said to be an equilibrium point of Eq. (1.2) if it is a fixed point of f.*

Graphically, an equilibrium point is the x-coordinate of the point where the function intersects the line $y = x$.

Definition 1.3.2 *Stability*

i) An equilibrium point $\bar{x}$ of Eq. (1.2) is called locally stable if, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq. (1.2) with

$$|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$$

then

$$|x_n - \bar{x}| < \epsilon, \ \textit{for all } n \geq 0.$$

ii) An equilibrium point $\bar{x}$ of Eq. (1.2) is called locally asymptotically stable if, $\bar{x}$ is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq. (1.2) with

$$|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + ... + |x_0 - \bar{x}| < \gamma,$$

then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

iii) An equilibrium point $\bar{x}$ of Eq. (1.2) is called a global attractor if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq. (1.2) we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

iv) An equilibrium point $\bar{x}$ of Eq. (1.2) is called globally asymptotically stable if $\bar{x}$ is locally stable , and $\bar{x}$ is globally attractor of Eq. (1.2).

v) An equilibrium point $\bar{x}$ of Eq. (1.2) is called unstable if $\bar{x}$ is not locally stable.

Let $\bar{x}$ be the equilibrium point of Eq. (1.2), and suppose that f is a continuously differentiable function in some neighborhood of $\bar{x}$.
Let the partial derivative of $f(u_0, u_1, ..., u_k)$ with respect to u_i be denoted as

$$p_i = \frac{\partial f}{\partial u_i}(\bar{x}, \bar{x}, ..., \bar{x}) \quad \text{for} \quad i = 0, 1, 2, ..., k.$$

Then the linearized equation of the difference equation around the equilibrium point is

$$z_{n+1} = p_0 z_n + p_1 z_{n-1} + ... + p_k z_{n-k}, \quad n = 0, 1, 2, ...$$

The characteristic equation of the difference equation about $\bar{x}$ is

$$\lambda^{k+1} - p_0\lambda^k - ... - p_{k-1}\lambda - p_k = 0. \tag{1.3}$$

The following theorem is known as the Linearized Stability Theorem.

Theorem 1.3.1 *[11]Linearized Stability Theorem*
Assume that f is a continuously differentiable function defined on some open neighborhood about the equilibrium point $\bar{x}$. Then the following statements are true:

i) If all roots of Eq. (1.3) have absolute value less than 1, then the equilibrium point $\bar{x}$ of Eq. (1.1) is locally asymptotically stable.

ii) If at least one of the roots of Eq.(1.3) has absolute value greater than 1, then the equilibrium point $\bar{x}$ is unstable.

iii) If all roots of Eq.(1.3) have absolute value greater than 1, then the equilibrium point $\bar{x}$ is a source.

1.4 Banach Spaces

To define the Banach space we must define the norm

Definition 1.4.1 *A norm on a linear space X is a function $\|.\| : X \to \mathbf{R}$, $\forall x \in X$ with the following properties:*

(a) $\|x\| > 0$, $\forall\ x \in X$, *and* $\|x\| = 0$ *if and only if* $x = 0$.

(b) $\|\lambda x\| = |\lambda|\|x\|$, $\forall x \in X$ *and* $\lambda \in \mathbf{R}$.

(c) $\|x + y\| \leq \|x\| + \|y\|$, $\forall\ x, y \in X$.

A *normed space* $(X, \|.\|)$ is a space X with a norm defined on it.

Definition 1.4.2 *[13] A metric, or distance function on the set X is*

$$d : X \times X \to \mathbf{R},$$

where

(a) $d(x, y) \geq 0$, $\forall x, y \in X$, *and* $d(x, y) = 0$ *if and only if* $x = y$.

(b) $d(x, y) = d(y, x)$, $\forall x, y \in X$.

(c) $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in X$.

A metric space (X, d) is a set X equipped with a metric d.
A metric space X is complete if every Cauchy sequence in X converges to a limit in X, and x_n is a Cauchy sequence if for every $\epsilon > 0$ there is a natural number $\mathbf{N}$ such that $|x_m - x_n| < \epsilon$, $\forall m, n \geq \mathbf{N}$.

Definition 1.4.3 *[13]**Banach Space***
Let $\mathbf{K}$ *be one of the fields* $\mathbf{R}$ *or* $\mathbf{C}$*, a Banach space over* $\mathbf{K}$ *is a normed* $\mathbf{K}$*-vector space* $(X, \|.\|)$ *with respect to the metric* $d(x, y) = \|x - y\|$*,* $x, y \in X$*.*

Definition 1.4.4 *[13] A space* X *is said to be* compact *if every open covering* $\mathbf{A}$ *of* X *contains a finite subcollection that also covers* X*.*

A set $S \subseteq \mathbf{R^n}$ is convex if and only if $\forall x, y \in S$ and $\lambda \in [0, 1]$;

$$\lambda x + (1 - \lambda)y \in S.$$

1.5 Periodic Points and Cycles

Definition 1.5.1 *[8] Let b be in the domain of f. Then:*

(i) b is called periodic point of f if for some positive integer k, $f^k(b) = b$. Hence a point is k-periodic if it is a fixed point of f^k, that is, if it is an equilibrium point of the difference equation

$$x(n+1) = g(x(n)),$$

where $g = f^k$.
The periodic orbit of b, $O(b) = \{b, f(b), f^2(b), ..., f^{k-1}(b)\}$, is often called a k-cycle.

(ii) b is called eventually k-periodic if for some positive integer m, $f^m(b)$ is a k-periodic point. In other words, b is eventually k-periodic if

$$f^{m+k}(b) = f^m(b).$$

Graphically, a k-periodic point is the x-coordinate of the point where the graph of f^k meets the diagonal line $y = x$.

Definition 1.5.2 *[8] Let b be a k-period point of f. Then b is:*

(i) stable if it is a stable fixed point of f^k.

(ii) asymptotically stable if it is an asymptotically stable fixed point of f^k.

(iii) unstable if it is an unstable fixed point of f^k.

The cycle $\{x(0) = b, x(1) = f(b), x(2) = f^2(b), ..., x(k-1) = f^{k-1}(b)\}$ is called a k-cycle.

1.6 Oscillating Sequences and Semicycles

Definition 1.6.1 *[7] A sequence $\{x_n\}$ is said to oscillate about zero or simply oscillate if the terms x_n are neither eventually all positive nor eventually all negative.*

Definition 1.6.2 *[7] A sequence $\{x_n\}$ is said to oscillate about $\{y_n\}$ if the sequence $\{x_n - y_n\}$ oscillates.*

Definition 1.6.3 *[7] Assume that $\{x_n\}$ and $\{y_n\}$ are positive sequences, we define a positive semicycle of $\{x_n\}$ relative to the sequence $\{y_n\}$ as a string of terms $C_+ = \{x_{l+1}, x_{l+2}, ..., x_m\}$ such that $x_i \geq y_i$ for $i = l+1, ..., m$ with $l \geq -1$ and $m \leq \infty$ and such that either $l = -1$ or $l \geq 0$ and $x_l < y_l$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} < y_{m+1}$.*

Definition 1.6.4 *[7] Let $\{x_n\}$, $\{y_n\}$ be two positive sequences, A negative semicycle of $\{x_n\}$ relative to $\{y_n\}$ is a string of terms $C_- = \{x_{j+1}, ..., x_l\}$, such that $x_i < y_i$ for $i = j+1, j+2, ..., l$, with $j \geq -1$ and $l \leq \infty$ and such that either $j = -1$ or $j \geq 0$ and $x_j \geq y_j$ and either $l = \infty$ or $l < \infty$ and $x_{l+1} \geq y_{l+1}$.*

1.7 Big o notation

Definition 1.7.1 *Let $f(x)$, $g(x)$ be two functions defined on R or C. Then we say that $f(x) = O(g(x))$, $x \to \infty$, if there is a positive constant M such that*

$$|f(x)| \leq M|g(x)| \text{ for all } \; x \geq x_0.$$

Proposition 1.7.1 *If the limit*

$$\lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = C < +\infty$$

then $f(x) = O(g(x))$.

$f = O(g)$ if f is of order not exceeding the order of g.
In order to use the big O notation, it is essential to understand how the O symbol behaves within a formula. Here we have a list of its properties

$$O(f(x)g(x)) = O(f(x))O(g(x))$$

$$O(f(x)) + O(g(x)) = O(|f(x)| + |g(x)|)$$

$$f(x) + O(g(x)) = O(|f(x)| + |g(x)|)$$

$$f(x)O(g(x)) = O(f(x)g(x))$$

$$O(cf(x)) = O(f(x)), \ \ c \in \mathbf{R} \ \ c \neq 0$$

Example
Show that

$$\left(\frac{n}{t^2+n^2}\right)^n = O(\frac{1}{t^n}) \ \ n \to \infty, \ \ \text{for } n \in Z^+$$

Solution
Without loss of generality we assume $t > 1$. We have that $t^2 + n^2 = (t-n)^2 + 2tn \geq 2nt$, then

$$\left(\frac{n}{t^2+n^2}\right)^n \leq \frac{1}{(2tn)^n} = \frac{1}{2^n}\left(\frac{1}{t^n}\right) \leq \frac{1}{t^n}.$$

Then

$$\left(\frac{n}{t^2+n^2}\right)^n = O(\frac{1}{t^n})$$

Using proposition (1.7.1) we can calculate the following examples

$$\begin{aligned} 6 &= O(1) \ \ \text{at any point} \\ 3x^2 &= O(x^2) \ \ \text{at any point} \\ \sin x &= O(1) \ \ \text{as} \ \ x \to 0 \\ \sin x &= O(x) \ \ \text{as} \ \ x \to 0. \end{aligned}$$

1.8 Taylor Series and polynomials

Definition 1.8.1 *Let f be a real function defined on a domain D. If the function is continuous at every point in D we say that it belongs to $C^0(D)$.*

If the function is differentiable n times at each point of D (excluding the boundary) and the $n-th$ derivative is continuous , we say that the function is in $C^n(D)$. If the function can be differentiated infinitely many times we say that it is in $C^\infty(D)$.

Theorem 1.8.1 *Taylor's Theorem*
If a function $f(x)$ belongs to $C^{n+1}(D)$ and $\alpha \in D$, then the function can be approximated with a degree n polynomial of this kind

$$P_{n.\alpha}(x) = \sum_{i=0}^{n} \frac{f^(i)(\alpha)}{i!}(x-\alpha)^i$$

Theorem 1.8.2 *Let f be a real function in $C^{n+1}([a,b])$, then for every $\alpha \in (a,b)$ there is a function $h_n(x)$ such that*

$$f(x) = P_{n,\alpha}(x) + h_n(x)(x-\alpha)^n$$

and

$$\lim_{x \to a} h_n(x) = 0$$

Theorem 1.8.3 *In the setup of the previous theorem, for every $x \in (a,b)$ there is a point η between x and α such that*

$$h_n(x)(x-\alpha)^n = \frac{f^{n+1}(\eta)}{(n+1)!}(x-a)^{n+1}.$$

Taylor Series
If a function (x) is $C^{+\infty}$ over some interval $[a,b]$, the Taylor series centered at some point $\alpha \in (a,b)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n$$

Remainder of Taylor polynomial as a big O

Proposition 1.8.1 *Let $f(x)$ be a function in $C^{n+1}([a,b])$ and α is a point in (a,b), the Taylor expansion can be written in big O notation :*

$$f(x) = P_{n,\alpha}(x) + O((x-\alpha)^{n+1})$$

For example,

$$e^x = 1 + x + \frac{x^2}{2} + O(x^3).$$

Chapter 2

On The Difference Equation $x_{n+1} = p_n + \frac{x_{n-1}}{x_n}$

Many authors studied the behavior of the difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \ \ n = 0, 1, ...,$$

where p_n is a positive bounded sequence and the initial values x_{-1},x_0 are positive and some of its extensions. We note that the papers [1],[7],[12],[20] were devoted for these equations.

In this part we are interested in studying boundedness, persistence, unbounded solutions, attractivity and the global asymptotic behavior of positive solutions of the nonautonomous difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \ \ n = 0, 1, ..., \tag{2.1}$$

with initial conditions $x_{-1} \geq 0$, $x_0 > 0$, and where $\{p_n\}$ is a positive bounded sequence, with

$$\liminf_{n\to\infty} p_n = p \geq 0 \ \text{ and } \ \limsup_{n\to\infty} p_n = q < \infty. \tag{2.2}$$

Theorem 2.0.4 *[7] Assume that all the roots of the polynomial*

$$P(t) = t^N - s_1 t^{N-1} - ... - s_N,$$

where $s_1, s_2, ..., s_N \geq 0$, have absolute value less than 1. If $\{x_n\}$ is a nonnegative solution of the inequality

$$x_{n+N} \leq s_1 x_{n+N-1} + ... + s_N x_n + y_n,$$

where $y_n \geq 0$, for $n = 0, 1, ...$, then the following statements are true:

(i) *If $\sum_{n=0}^{\infty} y_n$ converges, then $\sum_{n=0}^{\infty} x_n$ converges.*

(ii) *If $\{y_n\}$ is bounded, then $\{x_n\}$ is bounded.*

(iii) *If $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty} x_n = 0$.*

Theorem 2.0.5 *[7](Brower Fixed Point Theorem)*
The continuous operator

$$A : M \to M$$

has at least one fixed point when M is compact, convex, nonempty set in a finite dimensional normed space over $\mathbf{K}$($\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$).

2.1 Boundedness and persistence

We will dedicate this section for studying boundedness and persistence of Eq.(2.1) given Eq.(2.2).

Lemma 2.1.1 *[7] Let $\{x_n\}$ be a solution of (2.1), also assume that (2.2) is satisfied, then the following are true:*

(i) *If $p > 0$, then $\{x_n\}$ persists.*

(ii) *If $p > 1$, then $\{x_n\}$ is bounded from above.*

Proof. (i)It is obvious from the assumptions of Eq.(2.1) that $\{x_n\} > 0$ for all $n = -1, 0, ...$, this means that $\frac{x_{n-1}}{x_n} > 0$, which concludes that

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n} > p_n.$$

So we obtain $\liminf_{n\to\infty} x_n \geq \liminf_{n\to\infty} p_n = p > 0$, which implies the persistence of the sequence $\{x_n\}$.

(ii) In this part we aim to show that $\{x_n\}$ is bounded from above.
From part(i) we know that $x_n \geq p_{n-1} \geq p - \epsilon > 1$, for sufficiently large n, and for $\epsilon > 0$. Using Eq.(2.1) we get

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n} \leq p_n + \frac{x_{n-1}}{p - \epsilon}.$$

Referring to Theorem (2.0.4), $\{x_n\}$ is bounded since $\{p_n\}$ is bounded. ∎

Lemma 2.1.2 *[7] Assume that Eq.(2.2) is satisfied and $p > 1$, and let $\{x_n\}$ be a solution of Eq.(2.1). If*

$$\lambda = \liminf_{n\to\infty} x_n \ \ \textit{and} \ \ \mu = \limsup_{n\to\infty} x_n,$$

then

$$\frac{pq-1}{q-1} \leq \lambda \leq \mu \leq \frac{pq-1}{p-1}. \tag{2.3}$$

Proof. Let $\epsilon > 0$, then for $n \geq N_0(\epsilon)$, we have $\lambda - \epsilon \leq x_n \leq \mu + \epsilon$ and $p - \epsilon \leq p_n \leq q + \epsilon$. Then

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n} \geq p - \epsilon + \frac{\lambda - \epsilon}{\mu + \epsilon}, \tag{2.4}$$

and

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n} \leq q + \epsilon + \frac{\mu + \epsilon}{\lambda - \epsilon}. \tag{2.5}$$

As $n \to \infty$, we have

$$\lambda \geq p - \epsilon + \frac{\lambda - \epsilon}{\mu + \epsilon}, \tag{2.6}$$

and

$$\mu \leq q + \epsilon + \frac{\mu + \epsilon}{\lambda - \epsilon}. \tag{2.7}$$

It is known that $\epsilon > 0$ is arbitrary, hence,

$$\lambda \geq p + \frac{\lambda}{\mu},$$

and

$$\mu \leq q + \frac{\mu}{\lambda}.$$

Consequently,

$$\lambda\mu - p\mu \geq \lambda,$$

and

$$\lambda\mu - q\lambda \leq \mu.$$

So we get

$$\mu p + \lambda \leq \lambda\mu \leq q\lambda + \mu.$$

Hence,

$$\mu p - \mu \leq q\lambda - \lambda,$$

and so

$$\mu(p-1) \leq \lambda(q-1),$$

then

$$\frac{\mu}{\lambda} \leq \frac{q-1}{p-1} \text{ and } \frac{\lambda}{\mu} \geq \frac{p-1}{q-1}.$$

For $n > N_0$, from Eq.(2.4) and Eq.(2.5) and using Taylor's expansion at $\epsilon = 0$ we get

$$\begin{aligned} x_{n+1} &\geq p - \epsilon + \frac{\lambda - \epsilon}{\mu + \epsilon} \\ &= p + \frac{\lambda}{\mu} + O(\epsilon) \\ &\geq p + \frac{p-1}{q-1} + O(\epsilon) \\ &= \frac{pq - p + p - 1}{q-1} + O(\epsilon) \\ &= \frac{pq-1}{q-1} + O(\epsilon). \end{aligned}$$

To explain the calculations above we will use Taylor's expansion of the function $f(\epsilon) = p - \epsilon + \frac{\lambda-\epsilon}{\mu+\epsilon}$ centered at $\epsilon = 0$ which is

$$\begin{aligned} f(\epsilon) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n \\ &= p + \frac{\lambda}{\mu} + f^{'}(0)\epsilon + \frac{f^{''}}{2!}(0)\epsilon^2 + ... \\ &= p + \frac{\lambda}{\mu} + O(\epsilon), \end{aligned}$$

and

$$\begin{aligned}
x_{n+1} &\leq q + \epsilon + \frac{\mu + \epsilon}{\lambda - \epsilon} \\
&= q + \frac{\mu}{\lambda} + O(\epsilon) \\
&\leq q + \frac{q-1}{p-1} + O(\epsilon) \\
&= \frac{pq - q + q - 1}{p-1} + O(\epsilon) \\
&= \frac{pq-1}{p-1} + O(\epsilon).
\end{aligned}$$

Similarly,

$$\begin{aligned}
f(\epsilon) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n \\
&= q + \frac{\mu}{\lambda} + f^{'}(0)\epsilon + \frac{f^{''}}{2!}(0)\epsilon^2 + \ldots \\
&= q + \frac{\mu}{\lambda} + O(\epsilon).
\end{aligned}$$

$\epsilon > 0$ is arbitrary, as $n \to \infty$ we achieve the result which is

$$\lambda \geq \frac{pq-1}{q-1},$$

and

$$\mu \leq \frac{pq-1}{p-1}.$$

So we get

$$\frac{pq-1}{q-1} \leq \lambda \leq \mu \leq \frac{pq-1}{p-1}.$$

■

Theorem 2.1.1 *[7] Consider the interval* $I = [\frac{(PQ-1)}{(Q-1)}, \frac{(PQ-1)}{(P-1)}]$ *where* $1 < P \leq p_n \leq Q$*, for* $n = 0, 1, \ldots$*. If* $\{x_n\}$ *is a solution of Eq.(2.1) such that* x_{-1}*,* $x_0 \in I$*, then* $x_n \in I$*, for* $n = 0, 1, \ldots$*.*

Proof. We will use mathematical induction. Now,

$$x_1 = p_0 + \frac{x_{-1}}{x_0} \tag{2.8}$$

It was assumed that x_{-1}, $x_0 \in I = [\frac{(PQ-1)}{(Q-1)}, \frac{(PQ-1)}{(P-1)}]$ which concludes that

$$\frac{x_{-1}}{x_0} \leq \frac{\frac{(PQ-1)}{(P-1)}}{\frac{(PQ-1)}{(Q-1)}},$$

and

$$\frac{x_{-1}}{x_0} \geq \frac{\frac{(PQ-1)}{(Q-1)}}{\frac{(PQ-1)}{(P-1)}},$$

By substituting these two inequalities in (2.8) we get

$$\begin{aligned}
x_1 &= p_0 + \frac{x_{-1}}{x_0} \\
&\leq p_0 + \frac{\frac{(PQ-1)}{(P-1)}}{\frac{(PQ-1)}{(Q-1)}} \\
&\leq Q + \frac{\frac{(PQ-1)}{(P-1)}}{\frac{(PQ-1)}{(Q-1)}} \\
&= Q + \frac{Q-1}{P-1} \\
&= \frac{PQ-Q+Q-1}{P-1} \\
&= \frac{PQ-1}{P-1},
\end{aligned}$$

and

$$\begin{aligned}
x_1 &= p_0 + \frac{x_{-1}}{x_0} \\
&\geq p_0 + \frac{\frac{(PQ-1)}{(Q-1)}}{\frac{(PQ-1)}{(P-1)}} \\
&\geq P + \frac{\frac{(PQ-1)}{(Q-1)}}{\frac{(PQ-1)}{(P-1)}} \\
&= P + \frac{P-1}{Q-1} \\
&= \frac{PQ-P+P-1}{Q-1} \\
&= \frac{PQ-1}{Q-1}.
\end{aligned}$$

So $x_1 \in I$. Assume the result holds for $k = 2, 3, ..., n$. Now, we aim to prove the result for $k = n + 1$

$$\begin{aligned} x_{n+1} &= p_n + \frac{x_{n-1}}{x_n} \\ &\leq Q + \frac{\frac{(PQ-1)}{(P-1)}}{\frac{(PQ-1)}{(Q-1)}} \\ &= Q + \frac{Q-1}{P-1} \\ &= \frac{PQ - Q + Q - 1}{P-1} \\ &= \frac{PQ-1}{P-1}, \end{aligned}$$

and

$$\begin{aligned} x_{n+1} &= p_n + \frac{x_{n-1}}{x_n} \\ &\geq P + \frac{\frac{(PQ-1)}{(Q-1)}}{\frac{(PQ-1)}{(P-1)}} \\ &= P + \frac{P-1}{Q-1} \\ &= \frac{PQ - P + P - 1}{Q-1} \\ &= \frac{PQ-1}{Q-1}. \end{aligned}$$

So $x_{n+1} \in I$.
We conclude that $x_n \in I$, for all $n = 0, 1, ...$ ■

2.2 Existence of unbounded solutions

In this section we will introduce sufficient conditions for the existence of unbounded solutions of Eq.(2.1).

Lemma 2.2.1 *[7] Consider Eq.(2.1). Then the following are true:*

(i) Suppose there exists $0 < b < 1$ such that $0 < p_{2n+1} \leq b$. Choose $x_{-1} > \frac{1}{1-b}$ and $0 < x_0 < 1$. Then

$$x_{2n-1} > \frac{1}{1-b} \quad \textit{and} \quad 0 < x_{2n} < 1 \quad \textit{for all } n \geq 0.$$

(ii) Suppose there exists $0 < b < 1$ *such that* $0 < p_{2n} \leq b$*. Choose* $0 < x_{-1} < 1$ *and* $x_0 > \frac{1}{1-b}$*. Then*

$$0 < x_{2n-1} < 1 \ \textit{and} \ x_{2n} > \frac{1}{1-b} \ \textit{for all} \ n \geq 0.$$

Proof. To prove this lemma we will use mathematical induction.
(i)Assume that $x_{-1} > \frac{1}{1-b}$ and $0 < x_0 < 1$. Now,

$$x_1 = p_0 + \frac{x_{-1}}{x_0} > \frac{x_{-1}}{x_0} > \frac{\frac{1}{1-b}}{1} = \frac{1}{1-b},$$

and

$$0 < x_2 = p_1 + \frac{x_0}{x_1} < b + \frac{1}{\frac{1}{1-b}} = b + 1 - b = 1.$$

Assume that the result holds for all $k = 3, 4, ..., n-1$. In other words $x_{2k-1} > \frac{1}{1-b}$ and $0 < x_{2k} < 1$. Now, we need to show that the result is satisfied for $k = n$.

$$x_{2n-1} = p_{2n-2} + \frac{x_{2n-3}}{x_{2n-2}} > \frac{x_{2n-3}}{x_{2n-2}} > \frac{\frac{1}{1-b}}{1} > \frac{1}{1-b},$$

and

$$x_{2n} = p_{2n-1} + \frac{x_{2n-2}}{x_{2n-1}} < b + \frac{1}{\frac{1}{1-b}} = b + 1 - b = 1.$$

Consequently, $x_{2n-1} > \frac{1}{1-b}$ and $0 < x_{2n} < 1$, for all $n \geq 0$.
(ii)Assume that $0 < x_{-1} < 1$ and $x_0 > \frac{1}{1-b}$. Now,

$$0 < x_1 = p_0 + \frac{x_{-1}}{x_0} < b + \frac{x_{-1}}{x_0} < b + \frac{1}{\frac{1}{1-b}} = b + 1 - b = 1,$$

and

$$x_2 = p_1 + \frac{x_0}{x_1} > \frac{x_0}{x_1} > \frac{\frac{1}{1-b}}{1} = \frac{1}{1-b}.$$

Assume that for $k = 3, 4, ..., n-1$, $0 < x_{2k-1} < 1$ and $x_{2k} > \frac{1}{1-b}$. Now, for $k = n$

$$0 < x_{2n-1} = p_{2n-2} + \frac{x_{2n-3}}{x_{2n-2}} < b + \frac{1}{\frac{1}{1-b}} = b + 1 - b = 1,$$

and

$$x_{2n} = p_{2n-1} + \frac{x_{2n-2}}{x_{2n-1}} > \frac{x_{2n-2}}{x_{2n-1}} = \frac{\frac{1}{1-b}}{1} = \frac{1}{1-b}.$$

Then $0 < x_{2n-1} < 1$ and $x_{2n} > \frac{1}{1-b}$, for all $n \geq 0$. ∎

Lemma 2.2.2 *[7] Consider Eq.(2.1) and suppose that either*

$$0 < p_{2n+1} < 1 \ \text{and} \ \lim_{n\to\infty} p_{2n+1} = 0 \ \text{or} \ 0 < p_{2n} < 1 \ \text{and} \ \lim_{n\to\infty} p_{2n} = 0.$$

Then there exist unbounded solutions to Eq.(2.1).

Proof. We will use the mathematical induction to prove the result.
Case 1 Assume that

$$0 < p_{2n+1} < 1 \ \text{and} \ \lim_{n\to\infty} p_{2n+1} = 0.$$

Then there exists $0 < b < 1$ such that $p_{2n+1} \leq b$. Choose

$$x_{-1} > \frac{1}{1-b}, \ \text{and} \ 0 < x_0 < 1.$$

According to Lemma 2.2.1 we have

$$x_{2n-1} > \frac{1}{1-b} \ \text{and} \ 0 < x_{2n} < 1 \ \text{for all} \ n \geq 0.$$

Since $\lim_{n\to\infty} p_{2n+1} = 0$, there exists $N \geq 1$ such that $n \geq N-1$ and $p_{2n+1} < \frac{b}{2}$.

$$x_{2N} = p_{2N-1} + \frac{x_{2N-2}}{x_{2N-1}} < \frac{b}{2} + \frac{1}{\frac{1}{1-b}} = \frac{b}{2} + \frac{1-b}{1} = \frac{2-b}{2}.$$

$$x_{2N+1} = p_{2N} + \frac{x_{2N-1}}{x_{2N}} > \frac{x_{2N-1}}{x_{2N}} > \frac{\frac{1}{1-b}}{\frac{2-b}{2}} = \left(\frac{2}{2-b}\right)\frac{1}{1-b}.$$

$$x_{2N+2} = p_{2N+1} + \frac{x_{2N}}{x_{2N+1}} < \frac{b}{2} + \frac{1}{\frac{1}{1-b}} = \frac{b}{2} + \frac{1-b}{1} = \frac{2-b}{2}.$$

$$x_{2N+3} = p_{2N+2} + \frac{x_{2N+1}}{x_{2N+2}} > \frac{x_{2N+1}}{x_{2N+2}} > \frac{\left(\frac{2}{2-b}\right)\frac{1}{1-b}}{\frac{2-b}{2}} = \left(\frac{2}{2-b}\right)^2\frac{1}{1-b}.$$

$$x_{2N+4} = p_{2N+3} + \frac{x_{2N+2}}{x_{2N+3}} < \frac{b}{2} + \frac{1}{\frac{1}{1-b}} = \frac{b}{2} + \frac{1-b}{1} = \frac{2-b}{2}.$$

$$x_{2N+5} = p_{2N+4} + \frac{x_{2N+3}}{x_{2N+4}} > \frac{x_{2N+3}}{x_{2N+4}} > \frac{\left(\frac{2}{2-b}\right)^2\frac{1}{1-b}}{\frac{2-b}{2}} = \left(\frac{2}{2-b}\right)^3\frac{1}{1-b}.$$

Assume that for $n \geq N$

$$x_{2n} < \frac{2-b}{2} \ \text{and} \ x_{2n+1} > \left(\frac{2}{2-b}\right)^{n-N+1}\frac{1}{1-b}.$$

Now, for $n+1$ we have

$$x_{2n+2} = p_{2n+1} + \frac{x_{2n}}{x_{2n+1}} < \frac{b}{2} + \frac{1}{\frac{1}{1-b}} = \frac{b}{2} + \frac{1-b}{1} = \frac{2-b}{2},$$

and

$$x_{2n+3} = p_{2n+2} + \frac{x_{2n+1}}{x_{2n+2}} > \frac{x_{2n+1}}{x_{2n+2}} > \frac{\left(\frac{2}{2-b}\right)^{n-N+1}\frac{1}{1-b}}{\frac{2-b}{2}} = \left(\frac{2}{2-b}\right)^{n-N+2}\frac{1}{1-b}.$$

Then the solution is unbounded.

Case 2 Assume that

$$0 < p_{2n} < 1 \text{ and } \lim_{n\to\infty} p_{2n} = 0.$$

Then there exists $0 < b < 1$ such that $p_{2n} \leq b$. Choose

$$0 < x_{-1} < 1 \text{ and } x_0 > \frac{1}{1-b}.$$

According to Lemma 2.2.1 we have

$$0 < x_{2n-1} < 1 \text{ and } x_{2n} > \frac{1}{1-b} \text{ for all } n \geq 0.$$

Since $\lim_{n\to\infty} p_{2n} = 0$, there exists $N \geq 1$ such that for $n \geq N-1$ we have $p_{2n} < \frac{b}{2}$.

$$x_{2N} = p_{2N-1} + \frac{x_{2N-2}}{x_{2N-1}} > \frac{x_{2N-2}}{x_{2N-1}} > \frac{\frac{1}{1-b}}{1} = \frac{1}{1-b}.$$

$$x_{2N+1} = p_{2N} + \frac{x_{2N-1}}{x_{2N}} < \frac{b}{2} + \frac{1}{\frac{1}{1-b}} = \frac{b}{2} + 1 - b = \frac{2-b}{2}.$$

$$x_{2N+2} = p_{2N+1} + \frac{x_{2N}}{x_{2N+1}} > \frac{x_{2N}}{x_{2N+1}} > \frac{\frac{1}{1-b}}{\frac{2-b}{2}} = \left(\frac{2}{2-b}\right)\frac{1}{1-b}.$$

$$x_{2N+3} = p_{2N+2} + \frac{x_{2N+1}}{x_{2N+2}} < \frac{b}{2} + \frac{1}{\frac{1}{1-b}} = \frac{b}{2} + 1 - b = \frac{2-b}{2}.$$

$$x_{2N+4} = p_{2N+3} + \frac{x_{2N+2}}{x_{2N+3}} > \frac{x_{2N+2}}{x_{2N+3}} > \frac{\left(\frac{2}{2-b}\right)\frac{1}{1-b}}{\frac{2-b}{2}} = \left(\frac{2}{2-b}\right)^2\frac{1}{1-b}.$$

$$x_{2N+5} = p_{2N+4} + \frac{x_{2N+3}}{x_{2N+4}} < \frac{b}{2} + \frac{1}{\frac{1}{1-b}} = \frac{b}{2} + 1 - b = \frac{2-b}{2}.$$

$$x_{2N+6} = p_{2N+5} + \frac{x_{2N+4}}{x_{2N+5}} > \frac{x_{2N+4}}{x_{2N+5}} > \frac{\left(\frac{2}{2-b}\right)^2 \frac{1}{1-b}}{\frac{2-b}{2}} = \left(\frac{2}{2-b}\right)^3 \frac{1}{1-b}.$$

Assume that for $n \geq N$,

$$x_{2n} > \left(\frac{2}{2-b}\right)^{n-N} \frac{1}{1-b} \text{ and } x_{2n+1} < \frac{2-b}{2}.$$

Now, for $n+1$ we get

$$x_{2n+2} = p_{2n+1} + \frac{x_{2n}}{x_{2n+1}} > \frac{x_{2n}}{x_{2n+1}} > \frac{\left(\frac{2}{2-b}\right)^{n-N} \frac{1}{1-b}}{\frac{2-b}{2}} = \left(\frac{2}{2-b}\right)^{n-N+1} \frac{1}{1-b},$$

and

$$x_{2N+3} = p_{2N+2} + \frac{x_{2N+1}}{x_{2N+2}} < \frac{b}{2} + \frac{1}{\frac{1}{1-b}} = \frac{b}{2} + \frac{1-b}{1} = \frac{2-b}{2}.$$

Then this solution is unbounded. ■

Theorem 2.2.1 *[7] Suppose that* $0 < p_n < 1$ *and there exists* $0 < b < 1$ *such that for all* n *either*

$$p_{2n+1} \leq b \text{ or } p_{2n} \leq b.$$

Then there exist unbounded solutions to Eq.(2.1).

Proof. Case 1 $p_{2n+1} \leq b$
If $\sum_{n=0}^{\infty} p_{2n} < \infty$, then $\lim_{n\to\infty} p_{2n} = 0$, so there exist unbounded solutions according to Lemma 2.2.2.
If $\sum_{n=0}^{\infty} p_{2n} = \infty$, choose

$$x_{-1} > \frac{1}{1-b} \text{ and } 0 < x_0 < 1.$$

Referring to Lemma (2.2.1) $0 < x_{2n} < 1$ for all $n \geq 0$, and

$$x_1 = p_0 + \frac{x_{-1}}{x_0} > p_0 + \frac{\frac{1}{1-b}}{1} = p_0 + \frac{1}{1-b}.$$

$$x_3 = p_2 + \frac{x_1}{x_2} > p_2 + \frac{p_0 + \frac{1}{1-b}}{1} = p_2 + p_0 + \frac{1}{1-b}.$$

$$x_5 = p_4 + \frac{x_3}{x_4} > p_4 + \frac{p_2 + p_0 + \frac{1}{1-b}}{1} = p_4 + p_2 + p_0 + \frac{1}{1-b}.$$

Assume that for $k = n-1$,

$$x_{2n-1} > \sum_{h=0}^{n-1} p_{2h} + \frac{1}{1-b}.$$

We need to prove it for $k = n$. Now,

$$x_{2n+1} = p_{2n} + \frac{x_{2n-1}}{x_{2n}} > p_{2n} + \frac{\sum_{h=0}^{n-1} p_{2h} + \frac{1}{1-b}}{1} = \sum_{h=0}^{n} p_{2h} + \frac{1}{1-b}.$$

It is obvious that this subsequence is unbounded, so we have unbounded solutions to Eq.(2.1).

Case 2 $p_{2n} \leq b$

If $\sum_{n=0}^{\infty} p_{2n+1} < \infty$, then $\lim_{n\to\infty} p_{2n+1} = 0$, so there exist unbounded solutions according to Lemma 2.2.2.

If $\sum_{n=0}^{\infty} p_{2n+1} = \infty$, choose

$$0 < x_{-1} < 1 \text{ and } x_0 > \frac{1}{1-b}.$$

Referring to Lemma 2.2.1 we have that $0 < x_{2n+1} < 1$ for all $n \geq 0$.

$$x_2 = p_1 + \frac{x_0}{x_1} > p_1 + \frac{\frac{1}{1-b}}{1} = p_1 + \frac{1}{1-b}.$$

$$x_4 = p_3 + \frac{x_2}{x_3} > p_3 + \frac{p_1 + \frac{1}{1-b}}{1} = p_3 + p_1 + \frac{1}{1-b}.$$

$$x_6 = p_5 + \frac{x_4}{x_5} > p_5 + \frac{p_3 + p_1 + \frac{1}{1-b}}{1} = p_5 + p_3 + p_1 + \frac{1}{1-b}.$$

Assume that for $k = n-1$,

$$x_{2n-2} > \sum_{h=1}^{n-1} p_{2h-1} + \frac{1}{1-b}.$$

We need to prove this for $k = n$

$$x_{2n} = p_{2n-1} + \frac{x_{2n-2}}{x_{2n-1}} > p_{2n-1} + \frac{\sum_{h=1}^{n-1} p_{2h-1} + \frac{1}{1-b}}{1} = \sum_{h=1}^{n} p_{2h-1} + \frac{1}{1-b}.$$

It is clear that there exist unbounded solutions. ∎

2.3 Attractivity

In this section we will study the attractivity of Eq.(2.1). If $\bar{x}_n$ is a positive solution for Eq.(2.1) we are interested in finding sufficient conditions such that this solution attracts all the positive solutions x_n of the equation, which means that

$$x_n \to \bar{x}_n.$$

Let

$$y_n = \frac{x_n}{\bar{x}_n}, \text{ n} = -1, 0, 1, ...$$

From this we get

$$x_n = \bar{x}_n y_n.$$

Substituting this value in Eq.(2.1) we get

$$\bar{x}_{n+1} y_{n+1} = p_n + \frac{\bar{x}_{n-1} y_{n-1}}{\bar{x}_n y_n}.$$

$$y_{n+1} = \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}\frac{y_{n-1}}{y_n}}{\bar{x}_{n+1}} = \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}\frac{y_{n-1}}{y_n}}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}}.$$

So we get

$$y_{n+1} = \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}\frac{y_{n-1}}{y_n}}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}}. \tag{2.9}$$

Lemma 2.3.1 *[7] Let $\bar{x}_n$ be a positive solution of Eq.(2.1). Then the following are true.*

(i) Eq.(2.9) has a positive equilibrium solution $\bar{y} = 1$.

(ii) If for some n, $y_{n-1} < y_n$, then $y_{n+1} < 1$. Likewise, if for some n, $y_{n-1} \geq y_n$, then $y_{n+1} \geq 1$.

(iii) Every semicycle, except perhaps the first, of an oscillatory solution of Eq.(2.9) consists of exactly one term.

Proof. (i)

$$y = \frac{p_n + \frac{\bar{x}}{\bar{x}}\frac{y}{y}}{p_n + \frac{\bar{x}}{\bar{x}}} = 1.$$

So Eq.(2.9) has a positive equilibrium solution which is 1.
(ii) Assume that for some n, $y_{n-1} < y_n$ so $\frac{y_{n-1}}{y_n} < 1$. We need to show that $y_{n+1} < 1$. Now,

$$y_{n+1} = \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}\left(\frac{y_{n-1}}{y_n}\right)}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} < \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}(1)}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} = \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} = 1.$$

So $y_{n+1} < 1$.
In a similar method we can prove the second part. Assume that $y_{n-1} \geq y_n$, as a consequence $\frac{y_{n-1}}{y_n} \geq 1$. Now,

$$y_{n+1} = \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}\left(\frac{y_{n-1}}{y_n}\right)}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} \geq \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}(1)}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} = \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} = 1.$$

So $y_{n+1} \geq 1$.
(iii)We have two cases:
Case1:Assume that $y_{n-1} < 1$ and $y_n \geq 1$, then $\frac{y_{n-1}}{y_n} \leq 1$, which implies that

$$y_{n+1} = \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}\left(\frac{y_{n-1}}{y_n}\right)}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} \leq \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}(1)}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} = \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} = 1.$$

Consequently, the positive semicycle contains only one term.
Case2:Assume that $y_{n-1} > 1$ and $y_n \leq 1$, then $\frac{y_{n-1}}{y_n} \geq 1$, which implies that

$$y_{n+1} = \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}\left(\frac{y_{n-1}}{y_n}\right)}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} \geq \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}(1)}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} = \frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} = 1.$$

Consequently, the negative semicycle contains only one term.
According to the two cases every semicycle except possibly the first one consists of only one term. ∎

Lemma 2.3.2 *Every nonoscillatory solution to Eq.(2.9) converges to 1.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of Eq.(2.9). We have two possibilities either

$$y_n \leq 1 \text{ or } y_n > 1, \text{ for } n \geq N_0.$$

Without loss of generality assume that

$$y_n > 1 \text{ for } n \geq N_0.$$

Obviously $y_{n-1} > y_n$, for $n \geq N_0$. To prove this assume the contrary, in other words assume that there exist $k > N_0$ such that $y_{k-1} \leq y_k$, so $\frac{y_{k-1}}{y_k} \leq 1$, then we have

$$y_{k+1} = \frac{p_k + \frac{\bar{x}_{k-1}}{\bar{x}_k}\left(\frac{y_{k-1}}{y_k}\right)}{p_k + \frac{\bar{x}_{k-1}}{\bar{x}_k}} \leq \frac{p_k + \frac{\bar{x}_{k-1}}{\bar{x}_n}(1)}{p_k + \frac{\bar{x}_{k-1}}{\bar{x}_k}} = \frac{p_k + \frac{\bar{x}_{k-1}}{\bar{x}_k}}{p_k + \frac{\bar{x}_{k-1}}{\bar{x}_k}} = 1.$$

This gives that $y_{k+1} \leq 1$ which contradicts the assumption. As a result, $y_{n-1} > y_n$. It's clear that $\{y_n\}$ is decreasing and bounded below by 1, so it converges . Assume that $\lim_{n\to\infty} y_n = l$, we need to prove that $l = 1$.

$$\lim_{n\to\infty} \frac{y_{n-1}}{y_n} = \frac{\lim_{n\to\infty} y_{n-1}}{\lim_{n\to\infty} y_n} = \frac{l}{l} = 1.$$

Then, for $\epsilon > 0$ and for sufficiently large n we have

$$\left|\frac{y_{n-1}}{y_n} - 1\right| < \epsilon.$$

Then,

$$|y_{n+1} - 1| = \left|\frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}\frac{y_{n-1}}{y_n}}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}} - 1\right| = \left|\frac{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}\frac{y_{n-1}}{y_n} - p_n - \frac{\bar{x}_{n-1}}{\bar{x}_n}}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}}\right|$$

$$= \left|\frac{\frac{\bar{x}_{n-1}}{\bar{x}_n}\frac{y_{n-1}}{y_n} - \frac{\bar{x}_{n-1}}{\bar{x}_n}}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}}\right| = \left|\frac{\frac{\bar{x}_{n-1}}{\bar{x}_n}}{p_n + \frac{\bar{x}_{n-1}}{\bar{x}_n}}\right| \left|\frac{y_{n-1}}{y_n} - 1\right| < \left|\frac{y_{n-1}}{y_n} - 1\right| < \epsilon.$$

Now, for $n \geq N_0$

$$|y_{n+1} - 1| < \epsilon.$$

Consequently,

$$\lim_{n\to\infty} y_n = 1.$$

■

Theorem 2.3.1 *[7] Assume that*

$$p > 1 \;\; \text{and}\, q < p(p-1) + 1,$$

and let $\{\bar{x}\}$ be a particular positive solution of Eq.(2.1). Then for all positive solutions $\{x_n\}$ of Eq.(2.1),

$$x_n \to \bar{x}_n.$$

Proof. If $x_n \to \bar{x}_n$ then $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{x_n}{\bar{x}_n} = 1$, where $\{y_n\}$ satisfies Eq.(2.9). It is enough to show that $\lim_{n\to\infty} y_n = 1$. In this theorem we will focus on the case where $\{y_n\}$ oscillates about the equilibrium solution 1, since the other case was studied in the last Lemma.
Let's consider the function

$$g(p,t,s) = \frac{p+ts}{p+t}, \tag{2.10}$$

for p, s, $t > 0$.

$$\frac{\partial g}{\partial p} = \frac{(p+t)(1)-(p+ts)(1)}{(p+t)^2} = \frac{(t-ts)}{(p+t)^2} = \frac{t(1-s)}{(p+t)^2},$$

and

$$\frac{\partial g}{\partial t} = \frac{(p+t)s-(p+ts)(1)}{(p+t)^2} = \frac{ps+ts-p-ts}{(p+t)^2} = \frac{ps-p}{(p+t)^2} = \frac{p(s-1)}{(p+t)^2}.$$

From these derivatives we conclude that
(1)$g(p,t,s)$ is increasing in p for $s < 1$.
(2)$g(p,t,s)$ is decreasing in p for $s > 1$.
(3)$g(p,t,s)$ is increasing in t for $s > 1$.
(4)$g(p,t,s)$ is decreasing in t for $s < 1$.
Without loss of generality, there exists an integer N_0 such that

$$y_{2k} < 1 \text{ and } y_{2k+1} \geq 1 \text{ for } k \geq N_0.$$

This assumption is based on the fact that all semicycles excluding the first have only one term. Now, let

$$\gamma = \limsup_{n\to\infty} y_n \text{ and } \eta = \liminf_{n\to\infty} y_n.$$

From Eq.(2.9) and Eq.(2.10), we have

$$y_{2k+1} = g(p_{2k}, \frac{\bar{x}_{2k-1}}{\bar{x}_{2k}}, \frac{y_{2k-1}}{y_{2k}})$$

In addition, and with reference to Lemma (2.1.2), for $\epsilon > 0$ and k sufficiently large we have $\frac{y_{2k-1}}{y_{2k}} > 1$, $p_{2k} > p - \epsilon$ and $\frac{\bar{x}_{2k-1}}{\bar{x}_{2k}} \leq \frac{\mu+\epsilon}{\lambda-\epsilon}$, recalling that

$$\lambda = \liminf_{n\to\infty} x_n \text{ and } \mu = \limsup_{n\to\infty} x_n.$$

It is obvious that y_{2k+1} is increasing in $\frac{\bar{x}_{2k-1}}{\bar{x}_{2k}}$ and decreasing in p_{2k} when $\frac{y_{2k-1}}{y_{2k}} > 1$. Then,

$$\begin{aligned} y_{2k+1} &\le g(p-\epsilon, \frac{\mu+\epsilon}{\lambda-\epsilon}, \frac{y_{2k-1}}{y_{2k}}) \\ &= \frac{p-\epsilon+\frac{\mu+\epsilon}{\lambda-\epsilon}\frac{y_{2k-1}}{y_{2k}}}{p-\epsilon+\frac{\mu+\epsilon}{\lambda-\epsilon}} \\ &\le \frac{p-\epsilon+\frac{\mu+\epsilon}{\lambda-\epsilon}\frac{\gamma+\epsilon}{\eta-\epsilon}}{p-\epsilon+\frac{\mu+\epsilon}{\lambda-\epsilon}}. \end{aligned}$$

Hence,

$$\gamma = \limsup_{n\to\infty} y_{2k+1} \le \frac{p-\epsilon+\frac{\mu+\epsilon}{\lambda-\epsilon}\frac{\gamma+\epsilon}{\eta-\epsilon}}{p-\epsilon+\frac{\mu+\epsilon}{\lambda-\epsilon}}.$$

Depending on Lemma 2.1.2, it is true that $\frac{\mu}{\lambda} \le \frac{q-1}{p-1}$, $\epsilon > 0$ is arbitrary, so

$$\begin{aligned} \gamma &\le \frac{p+\frac{\mu}{\lambda}\frac{\gamma}{\eta}}{p+\frac{\mu}{\lambda}} \\ &\le \frac{p+\frac{q-1}{p-1}\frac{\gamma}{\eta}}{p+\frac{q-1}{p-1}} \\ &= \frac{\frac{p\eta}{\eta}+\frac{q-1}{p-1}\frac{\gamma}{\eta}}{p+\frac{q-1}{p-1}} \\ &= \frac{p\eta+\frac{q-1}{p-1}\gamma}{\eta\left(p+\frac{q-1}{p-1}\right)}. \end{aligned}$$

Then,

$$\gamma\eta \le \frac{p\eta+\frac{q-1}{p-1}\gamma}{p+\frac{q-1}{p-1}} = \frac{p\eta}{p+\frac{q-1}{p-1}} + \frac{\frac{q-1}{p-1}\gamma}{p+\frac{q-1}{p-1}}. \tag{2.11}$$

Similarly,

$$y_{2k+2} = g(p_{2k+1}, \frac{\bar{x}_{2k}}{\bar{x}_{2k+1}}, \frac{y_{2k}}{y_{2k+1}}),$$

for $\epsilon > 0$ and k sufficiently large we have $\frac{y_{2k}}{y_{2k+1}} < 1$, then y_{2k+2} is decreasing in $\frac{\bar{x}_{2k}}{\bar{x}_{2k+1}}$ and increasing in p_{2k+1}, also we have $p_{2k+1} > p-\epsilon$ and $\frac{\bar{x}_{2k}}{\bar{x}_{2k+1}} \le \frac{\mu+\epsilon}{\lambda-\epsilon}$.

Hence

$$
\begin{aligned}
y_{2k+2} &= \frac{p_{2k+1} + \frac{\bar{x}_{2k}}{\bar{x}_{2k+1}} \frac{y_{2k}}{y_{2k+1}}}{p_{2k+1} + \frac{\bar{x}_{2k}}{\bar{x}_{2k+1}}} \\
&\geq g(p-\epsilon, \frac{\mu+\epsilon}{\lambda-\epsilon}, \frac{y_{2k}}{y_{2k+1}}) \\
&= \frac{p-\epsilon + \frac{\mu+\epsilon}{\lambda-\epsilon} \frac{y_{2k}}{y_{2k+1}}}{p-\epsilon + \frac{\mu+\epsilon}{\lambda-\epsilon}} \\
&\geq \frac{p-\epsilon + \frac{\mu+\epsilon}{\lambda-\epsilon} \frac{\eta-\epsilon}{\gamma+\epsilon}}{p-\epsilon + \frac{\mu+\epsilon}{\lambda-\epsilon}}.
\end{aligned}
$$

$$
\eta = \liminf_{k\to\infty} y_{2k+1} \geq \frac{p-\epsilon + \frac{\mu+\epsilon}{\lambda-\epsilon} \frac{\eta-\epsilon}{\gamma+\epsilon}}{p-\epsilon + \frac{\mu+\epsilon}{\lambda-\epsilon}}.
$$

$\epsilon > 0$ is arbitrary and according to Lemma 2.1.2 $\frac{\mu}{\lambda} \leq \frac{q-1}{p-1}$, then

$$
\begin{aligned}
\eta &\geq \frac{p + \frac{q-1}{p-1} \frac{\eta}{\gamma}}{p + \frac{q-1}{p-1}} \\
&= \frac{\frac{p\gamma}{\gamma} + \frac{q-1}{p-1} \frac{\eta}{\gamma}}{p + \frac{q-1}{p-1}} \\
&= \frac{p\gamma + \frac{q-1}{p-1}\eta}{\gamma\left(p + \frac{q-1}{p-1}\right)}.
\end{aligned}
$$

Hence,

$$
\gamma\eta \geq \frac{p\gamma + \frac{q-1}{p-1}\eta}{p + \frac{q-1}{p-1}} = \frac{p\gamma}{p + \frac{q-1}{p-1}} + \frac{\frac{q-1}{p-1}\eta}{p + \frac{q-1}{p-1}}. \tag{2.12}
$$

From Eq.(2.11) and Eq.(2.12) we have

$$
\frac{p\gamma}{p + \frac{q-1}{p-1}} + \frac{\frac{q-1}{p-1}\eta}{p + \frac{q-1}{p-1}} \leq \gamma\eta \leq \frac{p\eta}{p + \frac{q-1}{p-1}} + \frac{\frac{q-1}{p-1}\gamma}{p + \frac{q-1}{p-1}}.
$$

Let

$$
a = \frac{p}{p + \frac{q-1}{p-1}}, \quad b = \frac{\frac{q-1}{p-1}}{p + \frac{q-1}{p-1}}.
$$

Hence,

$$
a\gamma + b\eta \leq \gamma\eta \leq a\eta + b\gamma. \tag{2.13}
$$

Then,

$$(a-b)\gamma \le (a-b)\eta$$

$$a-b=\frac{p}{p+\frac{q-1}{p-1}}-\frac{\frac{q-1}{p-1}}{p+\frac{q-1}{p-1}}=\frac{p-\frac{q-1}{p-1}}{p+\frac{q-1}{p-1}}=\frac{\frac{p(p-1)-q+1}{p-1}}{\frac{p(p-1)+q-1}{p-1}}=\frac{p(p-1)-q+1}{p(p-1)+q-1}>0,$$

since $p>1$ and $q<p(p-1)+1$, so $\gamma\le\eta$ and $\eta\le\gamma$. Hence $\gamma=\eta$ and $\lim_{n\to\infty} y_n=\lim_{n\to\infty}\frac{x_n}{\bar{x}_n}=1$. Consequently,

$$x_n \to \bar{x}_n.$$

■

2.4 Applications

In this section we aim to show some applications of the results discussed previously.

Definition 2.4.1 *[7] We say that $\{p_n\}$ is periodic with prime period k if k is the smallest integer such that*

$$p_{n+k}=p_n \ \textit{for} \ n=-1,0,....$$

Assume that $\{p_n\}$ is periodic with prime period k.

$$p=\liminf_{n\to\infty} p_n,$$

and

$$q=\limsup_{n\to\infty} p_n.$$

Lemma 2.4.1 *[7] A necessary condition for the existence of a periodic solution $\{x_n\}$ of Eq.(2.1) with prime period k is that $\{p_n\}$ is periodic with period k.*

Proof. Assume that the solution $\{x_n\}$ is a periodic solution with prime period k, this means that $x_{n+k}=x_n$, for $n=-1,0,....$

$$x_{n+k+1}=p_{n+k}+\frac{x_{n+k-1}}{x_{n+k}}.$$

Then,

$$p_{n+k} = x_{n+k+1} - \frac{x_{n+k-1}}{x_{n+k}} = x_{n+1} - \frac{x_{n-1}}{x_n} = p_n,$$

so we obtained that $p_{n+k} = p_n$, which means that p_n is periodic with period k. ■

Theorem 2.4.1 *[7] Assume that $\{p_n\}$ is periodic with prime period k, and let $1 < p < q$. Then the following statements are true:*

(i) There exists a positive periodic solution $\{\bar{x}_n\}$ of Eq.(2.1) with prime period k.

(ii) If $p > 1$ and $q < p(p-1)+1$, then the periodic solution $\{\bar{x}_n\}$ is unique and attracts all positive solutions of Eq.(2.1), that is,

$$\lim_{n\to\infty} \frac{x_n}{\bar{x}_n} = 1 \tag{2.14}$$

for all positive solutions $\{x_n\}$ of Eq.(2.1).

Proof. (i)We need to show that Eq.(2.1) has a periodic solution with period k, it suffices to show that the following system has a positive solution:

$$x_1 = p_k + \tfrac{x_{k-1}}{x_k}$$

$$x_2 = p_1 + \tfrac{x_k}{x_1}$$

$$x_3 = p_2 + \tfrac{x_1}{x_2}$$

$$\vdots$$

$$x_k = p_{k-1} + \tfrac{x_{k-2}}{x_{k-1}}$$

Define a function $F : R_+^k \to R_+^k$ such that

$$F(u_1, ..., u_k) = \left(p_k + \frac{u_{k-1}}{u_k}, p_1 + \frac{u_k}{u_1}, ..., p_{k-1} + \frac{u_{k-2}}{u_{k-1}}\right).$$

Also define an interval I such that $I = [\frac{pq-1}{q-1}, \frac{pq-1}{p-1}]$. Now, we need to show that I^k is invariant under the function F. If $u_1, ..., u_k \in I$, we have

$$\begin{aligned} p_i + \frac{u_j}{u_i} &\le q + \frac{\frac{pq-1}{p-1}}{\frac{pq-1}{q-1}} \\ &= q + \frac{q-1}{p-1} \\ &= \frac{pq-q+q-1}{p-1} \\ &= \frac{pq-1}{p-1}, \\ \text{for } \ i = 1, ..., k, \ j &= (i-1) \mod k, \\ \text{since the above system is periodic of period k,} \end{aligned}$$

and

$$\begin{aligned} p_i + \frac{u_j}{u_i} &\ge p + \frac{\frac{pq-1}{q-1}}{\frac{pq-1}{p-1}} \\ &= p + \frac{p-1}{q-1} \\ &= \frac{pq-p+p-1}{q-1} \\ &= \frac{pq-1}{q-1}, \\ \text{for } \ i = 1, ..., k, \ \text{for } \ j &= (i-1) \mod k. \\ \text{since the above system is periodic of period k.} \end{aligned}$$

So $p_i + \frac{u_j}{u_i} \in I$ for $\ i = 1, ..., k, \ j = (i-1) \mod k$. So I^k is invariant under the function F, in other words $F : I^k \to I^k$, it is obvious that F is continuous on I^k, and I^k is convex and compact set. Using Theorem 2.0.5, F has a fixed point in I^k.

Assume that the fixed point of F is $(\bar{u}_1, ..., \bar{u}_k) \in I^k$. Define the sequence $\{\bar{x}\}$ by

$$\bar{x}_{-1} = \bar{u}_{k-1}, \ \bar{x}_0 = \bar{u}_k \ \text{ and } \ \bar{x}_{mk+i} = \bar{u}_i, \ \text{ for } \ i = 1, ..., k, \ \ m = 0, 1, ...$$

It is clear that the sequence $\{\bar{x}_n\}$ satisfies Eq.(2.1) and is periodic with period k.

(ii)Assume that $p > 1$ and $q < p(p-1)+1$ and $\{p_n\}$ is periodic with prime period k, then

$$p = \liminf_{n\to\infty} p_n = \min_{1\le i\le k} \{p_i\}, \text{ and } q = \limsup_{n\to\infty} p_n = \max_{1\le i\le k} \{p_i\}.$$

According to Theorem 2.3.1, $\lim_{n\to\infty} \frac{x_n}{\bar{x}_n} = 1$ is satisfied for any solution $\{x_n\}$ of the Eq.(2.1), it remains to prove the uniqueness of the periodic solution $\{\bar{x}_n\}$. Assume that $\{y_n\}$ is another periodic solution of Eq.(2.1)with period k and different from $\{x_n\}$, since $\{y_n\}$ is periodic with period k then

$$y_{n+k} = y_n, \ \ n = -1, 0, 1, ...$$

Since the two solutions are different from each other there exists i such that

$$\frac{y_{nk+i}}{\bar{x}_{nk+i}} = \frac{y_i}{\bar{x}_i} \neq 1.$$

But this contradicts the conclusion of the theorem which states that $\lim_{n\to\infty} \frac{y_n}{\bar{x}_n} = 1$, then the solution is unique. ■

Corollary 2.4.1 *[7] Assume that $\{p_n\}$ is a convergent sequence and*

$$\lim_{n\to\infty} p_n = p > 1.$$

Then every solution $\{x_n\}$ of Eq.(2.1) is convergent and

$$\lim_{n\to\infty} x_n = p + 1.$$

Proof. $\{p_n\}$ is bounded so $\{x_n\}$ is bounded and persists according to Lemma 2.1.1.
Recalling that

$$\lambda = \liminf_{n\to\infty} x_n \text{ and } \mu = \limsup_{n\to\infty} x_n.$$

And

$$p = \liminf_{n\to\infty} p_n \text{ and } q = \limsup_{n\to\infty} p_n.$$

By Lemma 2.1.2,

$$\frac{pq-1}{q-1} \le \lambda \le \mu \le \frac{pq-1}{p-1}$$

Now, p_n is convergent so $p = \liminf_{n\to\infty} p_n = \limsup_{n\to\infty} p_n = q$. Then

$$p + 1 = \frac{p^2-1}{p-1} \le \lambda \le \mu \le \frac{p^2-1}{p-1} = p + 1$$

So we have that $\lambda = \mu = p + 1$. Hence, $\lim_{n\to\infty} x_n = p + 1$. ■

Chapter 3

On the Difference Equation $x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}$

This chapter is dedicated to study properties such as asymptotic behavior of the positive solutions, periodicity, and stability of equation

$$x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}, \;\; n = 0, 1, ..., \tag{3.1}$$

where A_n is a positive bounded sequence, the initial conditions x_{-1}, x_0 are positive constants, and p, $q \in (0, \infty)$.
The same equation were studied in papers [16], [17], [19].

3.1 Asymptotic behavior of the positive solutions

We aim in this section to find conditions so that if $\bar{x}_n$ is a fixed solution of the equation

$$x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}, \;\; n = 0, 1, ...,$$

then all solutions of (3.1) tend to the fixed solution $\bar{x}_n$. Let

$$y_n = \frac{x_n}{\bar{x}_n}, \quad n = -1, 0, 1, ... \tag{3.2}$$

Consequently,

$$y_{n+1} = \frac{x_{n+1}}{\bar{x}_{n+1}}, \quad n = -1, 0, 1, ...$$

From (3.1) we get

$$y_{n+1} = \frac{A_n + \frac{x_{n-1}^p}{x_n^q}}{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}.$$

Relation (3.2) gives that $x_{n-1} = \bar{x}_{n-1}y_{n-1}$ and $x_n = \bar{x}_n y_n$, then

$$y_{n+1} = \frac{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}\frac{y_{n-1}^p}{y_n^q}}{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}. \tag{3.3}$$

Lemma 3.1.1 *[16] Let y_n be a particular positive solution of (3.3).*
(a)*Suppose that there exists an $m \in \{0, 1, 2, ...\}$ such that*

$$y_{2m-1} \geq 1, \;\; y_{2m} < 1. \tag{3.4}$$

Then

$$y_{2n-1}^p > 1, \;\; y_{2n-1}^q > 1, \;\; y_{2n}^p < 1, \;\; y_{2n}^q < 1, \quad n = m+1, m+2, ... \tag{3.5}$$

(b)*Suppose that there exists an $m \in \{0, 1, 2, ...\}$ such that*

$$y_{2m-1} < 1, \;\; y_{2m} \geq 1. \tag{3.6}$$

Then

$$y_{2n-1}^p < 1, \;\; y_{2n-1}^q < 1, \;\; y_{2n}^p > 1, \;\; y_{2n}^q > 1, \quad n = m+1, m+2, ... \tag{3.7}$$

Proof. In both cases **(a)** and **(b)** we will use mathematical induction.
(a) We have the relation

$$y_{n+1} = \frac{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}\frac{y_{n-1}^p}{y_n^q}}{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}.$$

Now, for $n = m+1$ we have

$$y_{m+2} = \frac{A_{m+1} + \frac{\bar{x}_m^p}{\bar{x}_{m+1}^q}\frac{y_m^p}{y_{m+1}^q}}{A_{m+1} + \frac{\bar{x}_m^p}{\bar{x}_{m+1}^q}}. \tag{3.8}$$

If we replace m by $2m-1$, we get

$$y_{2m+1} = \frac{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}} \frac{y^p_{2m-1}}{y^q_{2m}}}{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}}}.$$

We assumed that $y_{2m-1} \geq 1$, $y_{2m} < 1$, and $y^p_{2m-1} \geq 1$, $y^q_{2m} < 1$ where p, q are positive constants, then $\frac{y^p_{2m-1}}{y^q_{2m}} > 1$. So

$$\begin{aligned} y_{2m+1} &= \frac{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}} \left(\frac{y^p_{2m-1}}{y^q_{2m}}\right)}{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}}} \\ &> \frac{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}}(1)}{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}}} \\ &= \frac{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}}}{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}}} = 1. \end{aligned}$$

So $y_{2m+1} > 1$, if we substitute $n-1$ in place of m we get $y_{2n-1} > 1$, which concludes that $y^q_{2n-1} > 1$ and $y^p_{2n-1} > 1$.
If we replace m by $2m$ in (3.8), we get

$$y_{2m+2} = \frac{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}} \frac{y^p_{2m}}{y^q_{2m+1}}}{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}}}.$$

We are given that $y_{2m} < 1$ and from above we have $y_{2m+1} > 1$, this implies that $y^p_{2m} < 1$ and $y^q_{2m+1} > 1$, as a result $\frac{y^p_{2m}}{y^q_{2m+1}} < 1$.
Thus

$$\begin{aligned} y_{2m+2} &= \frac{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}} \left(\frac{y^p_{2m}}{y^q_{2m+1}}\right)}{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}}} \\ &< \frac{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}}(1)}{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}}} \\ &= \frac{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}}}{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}}} = 1. \end{aligned}$$

As a result, $y_{2m+2} < 1$. It is known that $m = n - 1$, so we have $y_{2n} < 1$, which gives $y^p_{2n} < 1$, $y^q_{2n} < 1$.
For $n = m + 1$ we proved that

$$y^p_{2n-1} > 1, \;\; y^q_{2n-1} > 1, \;\; y^p_{2n} < 1, \;\; y^q_{2n} < 1.$$

Assume the result holds for $n = m+k-1$, where k is an integer greater than 2, from assumption we know that

$$y^p_{2(m+k-1)-1} > 1, \; y^q_{2(m+k-1)-1} > 1, \; y^p_{2(m+k-1)} < 1, \; y^q_{2(m+k-1)} < 1.$$

Which means that

$$y^p_{2(m+k)-3} > 1, \; y^q_{2(m+k)-3} > 1, \; y^p_{2(m+k)-2} < 1, \; y^q_{2(m+k)-2} < 1.$$

We need to show that the result holds for $n = m + k$. If we substitute $2(m + k) - 2$ in place of n in (3.3) we get

$$y_{2(m+k)-1} = \frac{A_{2(m+k)-2} + \frac{\bar{x}^p_{2(m+k)-3}}{\bar{x}^q_{2(m+k)-2}} \frac{y^p_{2(m+k)-3}}{y^q_{2(m+k)-2}}}{A_{2(m+k)-2} + \frac{\bar{x}^p_{2(m+k)-3}}{\bar{x}^q_{2(m+k)-2}}}.$$

From assumption we have $y^p_{2(m+k)-3} > 1$ and $y^q_{2(m+k)-2} < 1$
then $\frac{y^p_{2(m+k)-3}}{y^q_{2(m+k)-2}} > 1$.
Then

$$\begin{aligned} y_{2(m+k)-1} &= \frac{A_{2(m+k)-2} + \frac{\bar{x}^p_{2(m+k)-3}}{\bar{x}^q_{2(m+k)-2}} \left(\frac{y^p_{2(m+k)-3}}{y^q_{2(m+k)-2}}\right)}{A_{2(m+k)-2} + \frac{\bar{x}^p_{2(m+k)-3}}{\bar{x}^q_{2(m+k)-2}}} \\ &> \frac{A_{2(m+k)-2} + \frac{\bar{x}^p_{2(m+k)-3}}{\bar{x}^q_{2(m+k)-2}}(1)}{A_{2(m+k)-2} + \frac{\bar{x}^p_{2(m+k)-3}}{\bar{x}^q_{2(m+k)-2}}} = 1. \end{aligned}$$

So we conclude that $y^p_{2(m+k)-1} > 1$ and $y^q_{2(m+k)-1} > 1$, which gives that $y^p_{2n-1} > 1$ and $y^q_{2n-1} > 1$.
Now, if we replace n by $2(m + k) - 1$ in (3.3) we get the following equation

$$y_{2(m+k)} = \frac{A_{2(m+k)-1} + \frac{\bar{x}^p_{2(m+k)-2}}{\bar{x}^q_{2(m+k)-1}} \frac{y^p_{2(m+k)-2}}{y^q_{2(m+k)-1}}}{A_{2(m+k)-1} + \frac{\bar{x}^p_{2(m+k)-2}}{\bar{x}^q_{2(m+k)-1}}}.$$

From the assumption we have that $y^p_{2(m+k)-2} < 1$ and from the previous discussion we have $y^q_{2(m+k)-1} > 1$ so $\frac{y^p_{2(m+k)-2}}{y^q_{2(m+k)-1}} < 1$.
Then

$$\begin{aligned} y_{2(m+k)} &= \frac{A_{2(m+k)-1} + \frac{\bar{x}^p_{2(m+k)-2}}{\bar{x}^q_{2(m+k)-1}} \left(\frac{y^p_{2(m+k)-2}}{y^q_{2(m+k)-1}} \right)}{A_{2(m+k)-1} + \frac{\bar{x}^p_{2(m+k)-2}}{\bar{x}^q_{2(m+k)-1}}} \\ &< \frac{A_{2(m+k)-1} + \frac{\bar{x}^p_{2(m+k)-2}}{\bar{x}^q_{2(m+k)-1}}(1)}{A_{2(m+k)-1} + \frac{\bar{x}^p_{2(m+k)-2}}{\bar{x}^q_{2(m+k)-1}}} = 1. \end{aligned}$$

So $y_{2(m+k)} < 1$, as a result $y^p_{2(m+k)} < 1$ and $y^q_{2(m+k)} < 1$ and $y^p_{2n} < 1$, $y^q_{2n} < 1$ since $n = m + k$.
Then we conclude that

$$y^p_{2n-1} > 1, \;\; y^q_{2n-1} > 1, \;\; y^p_{2n} < 1, \;\; y^q_{2n} < 1, \;\; n = m+1, m+2, ...$$

(b) In a similar way we can prove the second part of this lemma. Assume (3.6). Now, replace n by $m + 1$ in the equation

$$y_{n+1} = \frac{A_n + \frac{\bar{x}^p_{n-1}}{\bar{x}^q_n} \frac{y^p_{n-1}}{y^q_n}}{A_n + \frac{\bar{x}^p_{n-1}}{\bar{x}^q_n}}.$$

To get

$$y_{m+2} = \frac{A_{m+1} + \frac{\bar{x}^p_m}{\bar{x}^q_{m+1}} \frac{y^p_m}{y^q_{m+1}}}{A_{m+1} + \frac{\bar{x}^p_m}{\bar{x}^q_{m+1}}}.$$

Now, replace m by $2m - 1$ to get

$$y_{2m+1} = \frac{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}} \frac{y^p_{2m-1}}{y^q_{2m}}}{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}}}.$$

We are given that $y_{2m-1} < 1$, $y_{2m} \geq 1$, which gives that $\frac{y^p_{2m-1}}{y^q_{2m}} < 1$. Now,

$$\begin{aligned} y_{2m+1} &= \frac{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}} \left(\frac{y^p_{2m-1}}{y^q_{2m}} \right)}{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}}} \\ &< \frac{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}}(1)}{A_{2m} + \frac{\bar{x}^p_{2m-1}}{\bar{x}^q_{2m}}} = 1. \end{aligned}$$

So $y_{2m+1} < 1$, substitute $n-1$ in place of m to obtain $y_{2n-1} < 1$, then $y^p_{2n-1} < 1$ and $y^q_{2n-1} < 1$.
If we replace m by $2m$ in (3.8) we get

$$y_{2m+2} = \frac{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}} \frac{y^p_{2m}}{y^q_{2m+1}}}{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}}}.$$

From the assumption we have that $y_{2m} \geq 1$ and it is true that $y_{2m+1} < 1$ from the previous discussion, so $\frac{y^p_{2m}}{y^q_{2m+1}} > 1$. Then we have

$$\begin{aligned} y_{2m+2} &= \frac{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}} \left(\frac{y^p_{2m}}{y^q_{2m+1}}\right)}{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}}} \\ &> \frac{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}}(1)}{A_{2m+1} + \frac{\bar{x}^p_{2m}}{\bar{x}^q_{2m+1}}} = 1. \end{aligned}$$

So we conclude that $y_{2m+2} > 1$, if we write this in terms of n we get $y_{2n} > 1$, which implies that $y^p_{2n} > 1$ and $y^q_{2n} > 1$.
So for $n = m+1$ we have

$$y^p_{2n-1} < 1, \;\; y^q_{2n-1} < 1, \;\; y^p_{2n} > 1, \;\; y^q_{2n} > 1.$$

Now, assume that the result holds for $n = m+k-1$, where $m+k-1$ is an integer and k is an integer greater than 2, so we have

$$y^p_{2(m+k-1)-1} < 1, \;\; y^q_{2(m+k-1)-1} < 1, \;\; y^p_{2(m+k-1)} > 1, \;\; y^q_{2(m+k-1)} > 1.$$

Which is equivalent to

$$y^p_{2(m+k)-3} < 1, \;\; y^q_{2(m+k)-3} < 1, \;\; y^p_{2(m+k)-2} > 1, \;\; y^q_{2(m+k)-2} > 1.$$

We aim to show that the result also holds for $n = m+k$, where $m+k$ is an integer.
Now, replace n by $2(m+k)-2$ in (3.3) to get

$$y_{2(m+k)-1} = \frac{A_{2(m+k)-2} + \frac{\bar{x}^p_{2(m+k)-3}}{\bar{x}^q_{2(m+k)-2}} \frac{y^p_{2(m+k)-3}}{y^q_{2(m+k)-2}}}{A_{2(m+k)-2} + \frac{\bar{x}^p_{2(m+k)-3}}{\bar{x}^q_{2(m+k)-2}}}.$$

According to the assumption it is true that $y^p_{2(m+k)-3} < 1$ and $y^q_{2(m+k)-2} > 1$ resulting in $\frac{y^p_{2(m+k)-3}}{y^q_{2(m+k)-2}} < 1$, then

$$y_{2(m+k)-1} = \frac{A_{2(m+k)-2} + \frac{\bar{x}^p_{2(m+k)-3}}{\bar{x}^q_{2(m+k)-2}}\left(\frac{y^p_{2(m+k)-3}}{y^q_{2(m+k)-2}}\right)}{A_{2(m+k)-2} + \frac{\bar{x}^p_{2(m+k)-3}}{\bar{x}^q_{2(m+k)-2}}} < \frac{A_{2(m+k)-2} + \frac{\bar{x}^p_{2(m+k)-3}}{\bar{x}^q_{2(m+k)-2}}(1)}{A_{2(m+k)-2} + \frac{\bar{x}^p_{2(m+k)-3}}{\bar{x}^q_{2(m+k)-2}}} = 1.$$

This implies that $y_{2n-1} < 1$ since $n = m+k$, also we conclude that $y^p_{2n-1} < 1$ and $y^q_{2n-1} < 1$.
If we take $n = 2(m+k)-1$ in (3.3) we get

$$y_{2(m+k)} = \frac{A_{2(m+k)-1} + \frac{\bar{x}^p_{2(m+k)-2}}{\bar{x}^q_{2(m+k)-1}}\frac{y^p_{2(m+k)-2}}{y^q_{2(m+k)-1}}}{A_{2(m+k)-1} + \frac{\bar{x}^p_{2(m+k)-2}}{\bar{x}^q_{2(m+k)-1}}}.$$

The assumption gives that $y^p_{2(m+k)-2} > 1$ and the previous discussion gives that $y^q_{2(m+k)-1} < 1$, so $\frac{y^p_{2(m+k)-2}}{y^q_{2(m+k)-1}} > 1$, then we have

$$y_{2(m+k)} = \frac{A_{2(m+k)-1} + \frac{\bar{x}^p_{2(m+k)-2}}{\bar{x}^q_{2(m+k)-1}}\left(\frac{y^p_{2(m+k)-2}}{y^q_{2(m+k)-1}}\right)}{A_{2(m+k)-1} + \frac{\bar{x}^p_{2(m+k)-2}}{\bar{x}^q_{2(m+k)-1}}} > \frac{A_{2(m+k)-1} + \frac{\bar{x}^p_{2(m+k)-2}}{\bar{x}^q_{2(m+k)-1}}(1)}{A_{2(m+k)-1} + \frac{\bar{x}^p_{2(m+k)-2}}{\bar{x}^q_{2(m+k)-1}}} = 1.$$

We conclude that $y_{2(m+k)} > 1$, in terms of n we have that $y_{2n} > 1$, then $y^p_{2n} > 1$ and $y^q_{2n} > 1$.
Then we conclude that

$$y^p_{2n-1} < 1,\ \ y^q_{2n-1} < 1,\ \ y^p_{2n} > 1,\ \ y^q_{2n} > 1,\ \ n = m+1, m+2, \ldots.$$

■

Lemma 3.1.2 *[16] Consider the function*

$$F(x,y,z) = \frac{z+xy}{z+x}, \quad x,y,z > 0. \tag{3.9}$$

Then the following statements are true:

(i) F is an increasing function in x for $y \in (1,\infty)$ and $z \in (0,\infty)$;

(ii) F is a decreasing function in x for $y \in (0,1)$;

(iii) F is an increasing function in y for any $x,z \in (0,\infty)$;

(iv) F is an increasing function in z for any $y \in (0,1)$ and $x \in (0,\infty)$;

(v) F is a decreasing function in z for any $y \in (1,\infty)$.

Proof. $F(x,y,z) = \frac{z+xy}{z+x}$.

$$\frac{\partial F}{\partial x} = \frac{z(y-1)}{(z+x)^2}.$$

$$\frac{\partial F}{\partial y} = \frac{x}{z+x}.$$

$$\frac{\partial F}{\partial z} = \frac{x(1-y)}{(z+x)^2}.$$

(i) $\frac{\partial F}{\partial x} = \frac{z(y-1)}{(z+x)^2} = 0$ if and only if $z = 0$ or $y = 1$ then F is increasing in x for $y \in (1,\infty)$ and $z \in (0,\infty)$.

(ii) It is obvious from (i) that F is decreasing in x for $y \in (0,1)$.

(iii) $\frac{\partial F}{\partial y} = \frac{x}{z+x} = 0$ if and only if $x = 0$ then F is increasing in y for $x,z \in (0,\infty)$.

(iv) $\frac{\partial F}{\partial z} = \frac{x(1-y)}{(z+x)^2} = 0$ if and only if $x = 0$ or $y = 1$ then F is increasing in z for $y \in (0,1)$ and $x \in (0,\infty)$.

(v) It is obvious from (iv) that F is decreasing in z for $y \in (1,\infty)$.

∎

Lemma 3.1.3 *[16] Suppose that A_n is a bounded sequence such that*

$$0 < m = \liminf_{n\to\infty} A_n, \quad M = \limsup_{n\to\infty} A_n < \infty. \tag{3.10}$$

Suppose also that

$$0 < p < 1. \tag{3.11}$$

Then every positive solution of Eq.(3.1) is bounded and persists.

Proposition 3.1.1 *[16] Consider Eq.(3.1) where A_n is bounded positive sequence such that (3.10) holds. Suppose also that*

$$0 < p + q < 1, \quad q > p. \tag{3.12}$$

Let $\bar{x}_n$ be a fixed solution of Eq.(3.1) and x_n be an arbitrary solution of Eq.(3.1). Then

$$\lim_{n\to\infty} y_n = 1, \tag{3.13}$$

where y_n is defined in (3.2).

Proof. Using Lemma(3.1.3) and relation(3.2)

$$0 < \eta = \liminf_{n\to\infty} y_n, \quad \theta = \limsup_{n\to\infty} y_n < \infty.$$

$$0 < k_1 = \liminf_{n\to\infty} \bar{x}_n, \quad k_2 = \limsup_{n\to\infty} \bar{x}_n < \infty. \tag{3.14}$$

We have two cases

Case 1 We suppose that there exists an $m \in \{1, 2, 3, ...\}$ such that either (3.4) or (3.6) holds. Assume that (3.4) holds. We obtain for $n \geq m$

$$y_{2n+1} = \frac{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}} \frac{y^p_{2n-1}}{y^q_{2n}}}{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}}, \quad y_{2n+2} = \frac{A_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}} \frac{y^p_{2n}}{y^q_{2n+1}}}{A_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}}}. \tag{3.15}$$

Let's consider the first equation

$$\begin{aligned}
y_{2n+1} &= \frac{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}} \frac{y^p_{2n-1}}{y^q_{2n}}}{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}} \\
&= \frac{A_{2n}\frac{y^q_{2n}}{y^q_{2n}} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}} \frac{y^p_{2n-1}}{y^q_{2n}}}{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}} \\
&= \frac{\frac{1}{y^q_{2n}}(A_{2n}y^q_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}} y^p_{2n-1})}{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}} \\
&= \frac{A_{2n}y^q_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}} y^p_{2n-1}}{y^q_{2n}(A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}})}. \\
y_{2n+1}y^q_{2n} &= \frac{A_{2n}y^q_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}} y^p_{2n-1}}{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}}.
\end{aligned}$$

We assumed that (3.4) is satisfied, as a result $y^p_{2n-1} > 1$, $y^q_{2n} < 1$ implying that $\frac{y^p_{2n-1}}{y^q_{2n}} > 1$.
By Lemma 3.1.2 y_{2n+1} is decreasing in A_n and increasing in $\frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}$, and so we have

$$\theta \leq \frac{m\eta^q + k\theta^p}{\eta^q(m+k)} \text{ then } \theta\eta^q \leq \frac{m\eta^q + k\theta^p}{m+k}, \quad k = \frac{k_2^p}{k_1^q}.$$

Now,

$$\begin{aligned}
y_{2n+2} &= \frac{A_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}} \frac{y^p_{2n}}{y^q_{2n+1}}}{A_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}}} \\
&= \frac{A_{2n+1}\frac{y^q_{2n+1}}{y^q_{2n+1}} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}} \frac{y^p_{2n}}{y^q_{2n+1}}}{A_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}}} \\
&= \frac{\frac{1}{y^q_{2n+1}}(A_{2n+1}y^q_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}} y^p_{2n})}{A_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}}} \\
&= \frac{A_{2n+1}y^q_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}} y^p_{2n}}{y^q_{2n+1}(A_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}})}.
\end{aligned}$$

We assumed that (3.4) holds, which means that $y^p_{2n-1} > 1$ and $y^q_{2n} < 1$, so $\frac{y^p_{2n-1}}{y^q_{2n}} > 1$, which implies that

$$y_{2n+1} = \frac{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}\left(\frac{y^p_{2n-1}}{y^q_{2n}}\right)}{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}} > \frac{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}(1)}{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}} = 1,$$

which gives that $y^q_{2n+1} > 1$, from the assumption we get that $y^p_{2n} < 1$ then $\frac{y^p_{2n}}{y^q_{2n+1}} < 1$. By Lemma 3.1.2 y_{2n+2} is increasing in A_{2n+1} and decreasing in $\frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}}$.
Thus we have

$$\eta \geq \frac{m\theta^q + k\eta^p}{\theta^q(m+k)} \text{ then } \eta\theta^q \geq \frac{m\theta^q + k\eta^p}{m+k},\ k = \frac{k_2^p}{k_1^q}.$$

Now,

$$\theta\eta^q \leq \frac{m\eta^q + k\theta^p}{m+k} \text{ gives } \theta\eta^q(m+k) \leq m\eta^q + k\theta^p.$$

And the equation

$$\eta\theta^q \geq \frac{m\theta^q + k\eta^p}{m+k} \text{ gives } \theta^q\eta(m+k) \geq m\theta^q + k\eta^p.$$

Hence

$$m + k \geq \frac{m\theta^q + k\eta^p}{\theta^q\eta}.$$

$$\theta\eta^q(m+k) \leq m\eta^q + k\theta^p.$$

Then

$$\theta\eta^q(\frac{m\theta^q + k\eta^p}{\theta^q\eta}) \leq m\eta^q + k\theta^p.$$

Thus,

$$\frac{\theta\eta^q m\theta^q}{\theta^q\eta} + \frac{\theta\eta^q k\eta^p}{\theta^q\eta} \leq m\eta^q + k\theta^p.$$

Consequently,

$$m\theta\eta^{q-1} + k\theta^{1-q}\eta^{p+q-1} \leq m\eta^q + k\theta^p.$$

Multiplying both sides of the preceding inequality by θ^{q-1} to get

$$m\theta^q\eta^{q-1} + k\eta^{p+q-1} \leq m\theta^{q-1}\eta^q + k\theta^{p+q-1}. \tag{3.16}$$

And this implies that

$$m\theta^q\eta^{q-1} - m\theta^{q-1}\eta^q \leq k\theta^{p+q-1} - k\eta^{p+q-1}.$$

$$m\theta^{q-1}\eta^{q-1}(\theta - \eta) \leq k(\eta\theta)^{p+q-1}(\eta^{1-p-q} - \theta^{1-p-q}).$$

Since $\eta \leq \theta$ and $p+q<1$, it is evident that

$$m\theta^{q-1}\eta^{q-1}(\theta - \eta) \leq k(\eta\theta)^{p+q-1}(\eta^{1-p-q} - \theta^{1-p-q}) \leq 0 \tag{3.17}$$

And so we have that $\eta = \theta$, in other words $\liminf\limits_{n\to\infty} y_n = \limsup\limits_{n\to\infty} y_n$, which implies that $\lim\limits_{n\to\infty} y_n$ exists.
Now, to determine the exact value of the limit, we have

$$y_{2n+1} = \frac{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}} \frac{y^p_{2n-1}}{y^q_{2n}}}{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}}$$

Using (3.5) $\frac{y^p_{2n-1}}{y^q_{2n}} > 1$, so

$$y_{2n+1} > \frac{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}(1)}{A_{2n} + \frac{\bar{x}^p_{2n-1}}{\bar{x}^q_{2n}}} = 1.$$

Thus as n goes to ∞, $\lim_{n\to\infty} y_n \geq 1$. Also we have

$$y_{2n+2} = \frac{A_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}} \frac{y^p_{2n}}{y^q_{2n+1}}}{A_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}}}$$

Using (3.5) $y^p_{2n} < 1$ and from above we have that $y^q_{2n+1} > 1$ so $\frac{y^p_{2n}}{y^q_{2n+1}} < 1$, then

$$y_{2n+2} < \frac{A_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}}(1)}{A_{2n+1} + \frac{\bar{x}^p_{2n}}{\bar{x}^q_{2n+1}}} = 1.$$

So as n goes to ∞, $\lim_{n\to\infty} y_n \leq 1$. We conclude that $\lim\limits_{n\to\infty} y_n = 1$.
The same procedure works if (3.6)holds.
Case 2 Suppose now neither (3.4) nor (3.6) holds. Now from Lemma 3.1.1 we get that

$$y_n < 1, \quad \text{or} \quad y_n \geq 1, \quad n \geq -1. \tag{3.18}$$

Without loss of generality assume that

$$y_n < 1, \quad n \geq -1. \tag{3.19}$$

Claim

$$y_{n+1}^q > y_n^p, \quad n \geq -1. \tag{3.20}$$

To prove this claim assume the contrary, in other words there exists a $\mu \geq -1$ such that

$$y_{\mu+1}^q \leq y_\mu^p. \tag{3.21}$$

$$y_{\mu+2} = \frac{A_{\mu+1} + \frac{\bar{x}_\mu^p}{\bar{x}_{\mu+1}^q}\frac{y_\mu^p}{y_{\mu+1}^q}}{A_{\mu+1} + \frac{\bar{x}_\mu^p}{\bar{x}_{\mu+1}^q}}.$$

Now, $\frac{y_\mu^p}{y_{\mu+1}^q} \geq 1$, which implies that

$$\begin{aligned} y_{\mu+2} &= \frac{A_{\mu+1} + \frac{\bar{x}_\mu^p}{\bar{x}_{\mu+1}^q}\frac{y_\mu^p}{y_{\mu+1}^q}}{A_{\mu+1} + \frac{\bar{x}_\mu^p}{\bar{x}_{\mu+1}^q}} \\ &\geq \frac{A_{\mu+1} + \frac{\bar{x}_\mu^p}{\bar{x}_{\mu+1}^q}(1)}{A_{\mu+1} + \frac{\bar{x}_\mu^p}{\bar{x}_{\mu+1}^q}} = 1. \end{aligned}$$

Therefore, $y_{\mu+2} \geq 1$, which contradicts the assumption in (3.19) so our claim is true. Now, we are given that $q > p$ in (3.12), and (3.19) gives that $y_n < 1$ for $n \geq -1$ so $y_n^p > y_n^q$. From this and (3.20) we get

$$y_{n+1}^q > y_n^p > y_n^q, \quad n \geq -1.$$

So

$$y_{n+1}^q > y_n^q, \quad n \geq -1.$$

As a result

$$y_{n+1} > y_n, \quad n \geq -1. \tag{3.22}$$

Now,

$$y_{n+1} = \frac{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}\frac{y_{n-1}^p}{y_n^q}}{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}.$$

By adding $\frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}$ and subtracting this expression from the numerator of the right hand side of the last equation we get

$$y_{n+1} = \frac{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}\frac{y_{n-1}^p}{y_n^q} + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q} - \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}.$$

$$y_{n+1} = \frac{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q} + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}\frac{y_{n-1}^p}{y_n^q} - \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}.$$

$$y_{n+1} = \frac{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}} + \frac{\frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}\frac{y_{n-1}^p}{y_n^q} - \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}.$$

$$y_{n+1} = 1 + \frac{\frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}\left(\frac{y_{n-1}^p}{y_n^q} - 1\right)}{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}.$$

Now,

$$|y_{n+1} - 1| = \left|\frac{\frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}\left(\frac{y_{n-1}^p}{y_n^q} - 1\right)}{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}\right|.$$

$$|y_{n+1} - 1| = \frac{\frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}\left(\left|\frac{y_{n-1}^p}{y_n^q} - 1\right|\right).$$

A_n is a positive sequence which implies that

$$|y_{n+1} - 1| < 1.\left|\frac{y_{n-1}^p}{y_n^q} - 1\right|.$$

Now, by (3.20) $\frac{y_{n-1}^p}{y_n^q} < 1$ and we are given that $y_{n+1} < 1$ then

$$1 - y_{n+1} < 1 - \frac{y_{n-1}^p}{y_n^q}, \text{ then } y_{n+1} > \frac{y_{n-1}^p}{y_n^q}. \tag{3.23}$$

According to (3.19) $y_{n+1} < 1$, as $n \to \infty$ we have $\lim_{n\to\infty} y_n \leq 1$, that is

$$\lim_{n\to\infty} y_n = \lambda \leq 1. \tag{3.24}$$

And we conclude from (3.23) that

$$y_{n+1}\left(\frac{y_n^q}{y_{n-1}^p}\right) = y_{n+1}y_n^q y_{n-1}^{-p} > 1.$$

As $n \to \infty$ we have

$$\lambda^{q-p+1} \geq 1. \tag{3.25}$$

Finally, we conclude from (3.24) and (3.25) that

$$\lambda = 1 \text{ so } \lim_{n\to\infty} y_n = 1.$$

∎

3.2 Periodicity and stability

Here we consider sufficient conditions for the existence and the uniqueness of 2-periodic and 3-periodic solutions for Eq.(3.1) and the convergence of the positive solutions of (3.1) to the periodic solutions.

Proposition 3.2.1 *[16] Consider Eq.(3.1). Then the following statements are true:*
(i) *Suppose that A_n is a positive two-periodic sequence such that*

$$A_{n+2} = A_n, \;\; n = 0, 1, 2, ... \tag{3.26}$$

Suppose also that $0 < p + q < 1$ and $p < q$. Then Eq.(3.1) has a unique two periodic solution and every positive solution of (3.1) tends to the unique two periodic solution.
(ii)*Suppose that A_n is a positive periodic sequence of period three such that*

$$A_{n+3} = A_n, \;\; n = 0, 1, 2, ... \tag{3.27}$$

Suppose also that $0 < p + q < 1$ and $p < q$ and there exist a positive number ϵ and a $\theta \in (0, \frac{1}{2}]$ such that

$$\frac{(B+\epsilon)^p}{C^q} < \epsilon, \;\; \frac{pq}{C^{2(q+1-p)}} + \frac{p\epsilon}{C} < \theta, \;\; \frac{p}{C^{q+1-p}} + \frac{q^2\epsilon}{C^{q+2-p}} < \theta, \tag{3.28}$$

where

$$B = \max\{A_0, A_1, A_2\}, C = \min\{A_0, A_1, A_2\}. \tag{3.29}$$

Then Eq.(3.1) has a unique periodic solution of period three and every positive solution of (3.1) tends to the unique three periodic solution.

Proof. **(i)** At the beginning, we show that (3.1) has a unique 2-periodic solution. Let x_n be a solution of (3.1). Now, x_n is two periodic if and only if the initial values x_{-1}, x_0 satisfy

$$x_{-1} = x_1 = A_0 + \frac{x_{-1}^p}{x_0^q}, \quad x_0 = x_2 = A_1 + \frac{x_0^p}{x_1^q}. \tag{3.30}$$

Let $x_{-1} = x$, $x_0 = y$ then from previous equation we get

$$x = A_0 + \frac{x^p}{y^q}, \quad y = A_1 + \frac{y^p}{x^q}. \tag{3.31}$$

We prove that (3.31) has a solution $(\bar{x}, \bar{y})$, $\bar{x} > 0$, $\bar{y} > 0$. From the first part of (3.31) we have that

$$\frac{x^p}{y^q} = x - A_0 \text{ this gives } \frac{y^q}{x^p} = \frac{1}{x - A_0}.$$

So we get

$$y = \frac{x^{\frac{p}{q}}}{(x - A_0)^{\frac{1}{q}}}. \tag{3.32}$$

From this and the second part of (3.31) we get

$$\begin{aligned} y - A_1 - \frac{y^p}{x^q} &= \frac{x^{\frac{p}{q}}}{(x - A_0)^{\frac{1}{q}}} - A_1 - \frac{\left(\frac{x^{\frac{p}{q}}}{(x-A_0)^{\frac{1}{q}}}\right)^p}{x^q} \\ &= \frac{x^{\frac{p}{q}}}{(x - A_0)^{\frac{1}{q}}} - A_1 - \frac{x^{\frac{p^2}{q}} x^{-q}}{(x - A_0)^{\frac{p}{q}}} = 0. \end{aligned}$$

So we have

$$\frac{x^{\frac{p}{q}}}{(x - A_0)^{\frac{1}{q}}} - A_1 - \frac{x^{\frac{p^2-q^2}{q}}}{(x - A_0)^{\frac{p}{q}}} = 0. \tag{3.33}$$

Consider the function

$$f(x) = \frac{x^{\frac{p}{q}}}{(x - A_0)^{\frac{1}{q}}} - A_1 - \frac{x^{\frac{p^2-q^2}{q}}}{(x - A_0)^{\frac{p}{q}}}. \tag{3.34}$$

Write $f(x)$ as

$$f(x) = \frac{1}{(x - A_0)^{\frac{p}{q}}} \left(\frac{x^{\frac{p}{q}}}{(x - A_0)^{\frac{1-p}{q}}} - x^{\frac{p^2-q^2}{q}} \right) - A_1. \tag{3.35}$$

Then

$$\lim_{x\to A_0} f(x)=\infty,\ \lim_{x\to\infty} f(x)=-A_1. \tag{3.36}$$

So Eq.(3.34) has a solution $\bar{x}>A_0$. Then if

$$\bar{y}=\frac{\bar{x}^{\frac{p}{q}}}{(\bar{x}-A_0)^{\frac{1}{q}}}$$

we find that the solution $\bar{x}_n$ of (3.1) with initial values $x_{-1}=\bar{x}$, $x_0=\bar{y}$ is a periodic solution of period two.

Finally, using proposition 3.1.1 it is clear that x_n is the unique periodic solution of period two and every positive solution of (3.1) tends to the unique periodic solution of period two and this is obvious since $y_n=\frac{x_n}{\bar{x}_n}$ and $\lim\limits_{n\to\infty} y_n = 1$ then $x_n\to\bar{x}_n$.

(ii) x_n is a three-periodic solution of (3.1) if

$$x_2=x_{-1}=A_1+\frac{x_0^p}{x_1^q},\quad x_3=x_0=A_2+\frac{x_1^p}{x_{-1}^q}. \tag{3.37}$$

We set $x_{-1}=x$, $x_0=y$ in (3.37) and we consider the system of nonlinear difference equations

$$x=A_1+\frac{y^p}{(h(x,y))^q},\ \ y=A_2+\frac{(h(x,y))^p}{x^q}, \tag{3.38}$$

where $h(x,y)$=x_1=$A_0+\frac{x_{-1}^p}{x_0^q}$=$A_0+\frac{x^p}{y^q}$. We consider the function

$$H:[A_1,A_1+\epsilon]\times[A_2,A_2+\epsilon]\to\mathbf{R},$$

such that

$$H(x,y)=(f(x,y),g(x,y)),\ \ f(x,y)=A_1+\frac{y^p}{(h(x,y))^q},\ \ g(x,y)=A_2+\frac{(h(x,y))^p}{x^q}. \tag{3.39}$$

First we prove that the function H is in $[A_1,A_1+\epsilon]\times[A_2,A_2+\epsilon]$.

It is obvious that for all $(x,y)\in[A_1,A_1+\epsilon]\times[A_2,A_2+\epsilon]$

$$f(x,y)=A_1+\frac{y^p}{(h(x,y))^q}>A_1,\ since\ \frac{y^p}{(h(x,y))^q}>0, \tag{3.40}$$

also we have

$$g(x,y)=A_2+\frac{(h(x,y))^p}{x^q}>A_2,\ since\ \frac{(h(x,y))^p}{x^q}>0. \tag{3.41}$$

Moreover, from(3.28), (3.29) and (3.38) and since $\frac{x^p}{y^q} > 0$ we get for $(x, y) \in [A_1, A_1 + \epsilon] \times [A_2, A_2 + \epsilon]$,

$$\begin{aligned} f(x,y) &= A_1 + \frac{y^p}{(h(x,y))^q} \\ &= A_1 + \frac{y^p}{\left(A_0 + \frac{x^p}{y^q}\right)^q} \\ &\leq A_1 + \frac{(A_2+\epsilon)^p}{\left(A_0 + \frac{x^p}{y^q}\right)^q} \\ &\leq A_1 + \frac{(A_2+\epsilon)^p}{A_0^q} \\ &\leq A_1 + \frac{(B+\epsilon)^p}{C^q} \\ &< A_1 + \epsilon. \end{aligned}$$

$$\begin{aligned} g(x,y) &= A_2 + \frac{(h(x,y))^p}{x^q} \\ &= A_2 + \frac{(A_0 + \frac{x^p}{y^q}))^p}{x^q} \\ &\leq A_2 + \frac{\left(A_0 + \frac{(A_1+\epsilon)^p}{A_2^q}\right)^p}{A_1^q} \\ &\leq A_2 + \frac{\left(B + \frac{(B+\epsilon)^p}{C^q}\right)^p}{C^q} \\ &< A_2 + \frac{(B+\epsilon)^p}{C^q} \\ &< A_2 + \epsilon. \end{aligned}$$

So we have that

$$f(x,y) < A_1 + \epsilon. \tag{3.42}$$

$$g(x,y) < A_2 + \epsilon. \tag{3.43}$$

Which implies that the function H is in $[A_1, A_1 + \epsilon] \times [A_2, A_2 + \epsilon]$ as needed.

Now, we need to show that the function H is contraction in $[A_1, A_1+\epsilon] \times [A_2, A_2+\epsilon]$.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{(h(x,y))^q \times 0 - qy^q(h(x,y))^{q-1}\frac{\partial h(x,y)}{\partial x}}{(h(x,y))^{2q}} \\ &= \frac{-yq(h(x,y))^{q-1}}{(h(x,y))^{2q}}\frac{py^q x^{p-1}}{y^{2q}} \\ &= \frac{-pq}{y^{q-p}x^{1-p}(h(x,y))^{q+1}}.\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{(h(x,y))^q py^{p-1} - qy^p(h(x,y))^{q-1}\frac{\partial h(x,y)}{\partial y}}{(h(x,y))^{2q}} \\ &= \frac{py^{p-1}h(x,y)^q + qy^p h(x,y)^{q-1}\frac{qx^p y^{q-1}}{y^{2q}}}{(h(x,y))^{2q}} \\ &= \frac{py^{p-1}h(x,y)^q + q^2x^p y^{p-q-1}(h(x,y))^{q-1}}{(h(x,y))^{2q}} \\ &= \frac{p}{y^{1-p}(h(x,y))^q} + \frac{q^2x^p}{y^{q-p+1}(h(x,y))^{q+1}}.\end{aligned}$$

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{px^q(h(x,y))^{p-1}\frac{\partial h(x,y)}{\partial x} - qx^{q-1}(h(x,y))^p}{x^{2q}} \\ &= \frac{px^q(h(x,y))^{p-1}[\frac{py^q x^{p-1}}{y^{2q}}] - qx^{q-1}(h(x,y))^p}{x^{2q}} \\ &= \frac{p^2(h(x,y))^{p-1}}{y^q x^{q-p+1}} - \frac{q(h(x,y))^p}{x^{q+1}}.\end{aligned}$$

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{px^q(h(x,y))^{p-1}\frac{\partial h(x,y)}{\partial y} - 0}{x^{2q}} \\ &= \frac{-pqx^{p+q}y^{p-1}(h(x,y))^{p-1}}{x^{2q}y^{2q}} \\ &= \frac{-pq}{x^{q-p}y^{q+1}(h(x,y))^{1-p}}.\end{aligned}$$

We will use

$$\frac{pq}{C^{2(q+1-p)}} < \frac{pq}{C^{2(q+1-p)}} + \frac{q\epsilon}{C} < \theta, \text{ since } \frac{q\epsilon}{C} > 0.$$

Now,

$$\begin{aligned}\left|\frac{\partial f}{\partial x}\right| &= \left|\frac{-pq}{y^{q-p}x^{1-p}(h(x,y))^{q+1}}\right| \\ &< \frac{pq}{C^{q-p}C^{1-p}C^{q+1}} \\ &= \frac{pq}{C^{2q-2p+2}} \\ &= \frac{pq}{C^{2(q-p+1)}} < \theta.\end{aligned}$$

$$\begin{aligned}\left|\frac{\partial f}{\partial y}\right| &= \left|\frac{p}{y^{1-p}(h(x,y))^{q}} + \frac{q^2x^p}{y^{q-p+1}(h(x,y))^{q+1}}\right| \\ &< \frac{p}{C^{1-p}C^{q}} + \frac{q^2(A_1+\epsilon)^p}{C^{q-p+1}C^{q+1}} \\ &< \frac{p}{C^{q-p+1}} + \frac{q^2\frac{(B+\epsilon)^p}{C^q}}{C^{q-p+2}} \\ &< \frac{p}{C^{q-p+1}} + \frac{q^2\epsilon}{C^{q-p+2}} < \theta.\end{aligned}$$

$$\begin{aligned}\left|\frac{\partial g}{\partial x}\right| &= \left|\frac{p^2(h(x,y))^{p-1}}{y^q x^{q-p+1}} - \frac{q(h(x,y))^p}{x^{q+1}}\right| \\ &< \frac{q\left(A_0+\frac{(A_1+\epsilon)^p}{A_2^q}\right)^p}{C^{q+1}} + \frac{p^2}{C^{2q-p+1}A_0^{1-p}} \\ &< \frac{q\left(B+\frac{(B+\epsilon)^p}{C^q}\right)^p}{C^{q+1}} + \frac{p^2}{C^{2(q-p+1)}} \\ &< \frac{q(B+\epsilon)^p}{C^{q+1}} + \frac{p^2}{C^{2(q-p+1)}} \\ &< \frac{\epsilon q}{C} + \frac{p^2}{C^{2(q-p+1)}} < \theta.\end{aligned}$$

$$\begin{aligned}\left|\frac{\partial g}{\partial y}\right| &= \left|\frac{-pq}{x^{p-q}y^{q+1}(h(x,y))^{1-p}}\right| \\ &< \frac{pq}{C^{2q-p+1}A_0^{1-p}} \\ &< \frac{pq}{C^{2(q-p+1)}} < \theta.\end{aligned}$$

So we conclude that

$$\left|\frac{\partial f}{\partial x}\right| < \theta, \quad \left|\frac{\partial f}{\partial y}\right| < \theta, \quad \left|\frac{\partial g}{\partial x}\right| < \theta, \quad \left|\frac{\partial g}{\partial y}\right| < \theta. \tag{3.44}$$

Moreover, there exist $\xi_i \in [A_1, A_1 + \epsilon]$, $\eta_i \in [A_2, A_2 + \epsilon]$, $i = 1, 2$ such that for all $x_1, x_2 \in [A_1, A_1 + \epsilon]$ and $y_1, y_2 \in [A_2, A_2 + \epsilon]$.

$$\begin{aligned}
f(x_1, y_1) - f(x_1, y_2) &= \frac{\partial f(x_1, \eta_1)}{\partial y}(y_1 - y_2), \\
f(x_1, y_2) - f(x_2, y_2) &= \frac{\partial f(\xi_1, y_2)}{\partial x}(x_1 - x_2), \\
g(x_1, y_1) - g(x_1, y_2) &= \frac{\partial g(x_1, \eta_2)}{\partial y}(y_1 - y_2), \\
g(x_1, y_2) - g(x_2, y_2) &= \frac{\partial g(\xi_2, y_2)}{\partial x}(x_1 - x_2).
\end{aligned} \tag{3.45}$$

$$\begin{aligned}
|f(x_1, y_1) - f(x_2, y_2)| &= |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\le |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)| \\
&\le 2\theta \max\{|x_1 - x_2|, |y_1 - y_2|\}.
\end{aligned}$$

And

$$\begin{aligned}
|g(x_1, y_1) - g(x_2, y_2)| &= |g(x_1, y_1) - g(x_1, y_2) + g(x_1, y_2) - g(x_2, y_2)| \\
&\le |g(x_1, y_1) - g(x_1, y_2)| + |g(x_1, y_2) - g(x_2, y_2)| \\
&\le 2\theta \max\{|x_1 - x_2|, |y_1 - y_2|\}.
\end{aligned}$$

Thus

$$\max\{|f(x_1, y_1) - f(x_2, y_2)|, |g(x_1, y_1) - g(x_2, y_2)|\} \le 2\theta \max\{|x_1 - x_2|, |y_1 - y_2|\} \tag{3.46}$$

Definition 3.2.1 *Let (X, d) be a complete metric space. A function $f : X \to X$ is called contraction if there exists $k < 1$ such that for any x, $y \in X$*

$$d(f(x), f(y)) \le kd(x, y).$$

Now, from(3.46) and since $\theta \in (0, \frac{1}{2})$ the function H is contraction in $[A_1, A_1 + \epsilon] \times [A_2, A_2 + \epsilon]$.

Theorem 3.2.1 *Banach Contraction Principle*
If $f : X \to X$ is a mapping on a complete metric space (X, d) into itself, and there exists a number $\alpha < 1$ such that for any two points x, $y \in X$

$$d(f(x), f(y)) \leq \alpha d(x, y).$$

Then, f has a unique fixed point, and for any x in X the sequence $f^n(x)$ converges to some point.

Hence, according to Banach Contraction Principle there exist a unique $(\bar{x}, \bar{y}) \in [A_1, A_1 + \epsilon] \times [A_2, A_2 + \epsilon]$ such that

$$\bar{x} = f(\bar{x}, \bar{y}), \;\; \bar{y} = g(\bar{x}, \bar{y}).$$

Therefore the solution x_n with initial values $x_{-1} = \bar{x}$, $x_0 = \bar{y}$ is periodic solution of period three. Using proposition (3.1.1) it is obvious that x_n is the unique solution of period three and every positive solution of (3.1) tends to the unique 3-periodic solution of (3.1) as $n \to \infty$. ■

Proposition 3.2.2 *[16] Consider Eq.(3.1)and assume that $0 < p + q < 1$, $p < q$. Then the following statements are true:*
(i) *Suppose that*

$$A_{n+2} = A_n, \;\; n = 0, 1, 2, ...$$

Suppose also that

$$\frac{p}{A_1^q A_0^{1-p}} + \frac{p^2 + q^2}{(A_1 A_0)^{q+1-p}} + \frac{p}{A_0^q A_1^{1-p}} < 1. \tag{3.47}$$

Then the unique 2-periodic solution of (3.1) is globally asymptotically stable.
(ii) *Assume that*

$$A_{n+3} = A_n, \;\; n = 0, 1, 2, ...$$

And

$$\frac{(B + \epsilon)^p}{C^q} < \epsilon, \;\; \frac{pq}{C^{2(q+1-p)}} + \frac{p\epsilon}{C} < \theta, \;\; \frac{p}{C^{q+1-p}} + \frac{q^2 \epsilon}{C^{q+2-p}} < \theta,$$

where

$$B = \max\{A_0, A_1, A_2\}, C = \min\{A_0, A_1, A_2\}.$$

Suppose also that

$$\frac{3pq}{C^{2(p+q-1)}} + \frac{p^3+q^3}{C^{3(p+q-1)}} < 1. \tag{3.48}$$

Then the unique 3-periodic solution of (3.1) is globally asymptotically stable.

Proof. **(i)** From proposition(3.2.1) there exists a unique periodic solution $\bar{x}_n$ of period two.
Let

$$x_{2n-1} = \bar{x}, \;\; x_{2n} = \bar{y}, \;\; n = 0, 1, 2, ...$$

We have

$$x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}.$$

Since the solution is two periodic we have that

$$x_{-1} = x_1 = x_3 = ... = x_{2n+1}, \;\; n = -1, 0, 1, 2, ...,$$

and

$$x_0 = x_2 = x_4 = ... = x_{2n}, \;\; n = 0, 1, 2, ...$$

Consequently,

$$x_{2n+1} = A_0 + \frac{x_{2n-1}^p}{x_{2n}^q}, \; x_{2n+2} = A_1 + \frac{x_{2n}^p}{x_{2n+1}^q}. \tag{3.49}$$

Then

$$x_{2n+1} = A_0 + \frac{\bar{x}^p}{\bar{y}^q}, \;\; x_{2n+2} = A_1 + \frac{\bar{y}^p}{\bar{x}^q}.$$

Now, if we set $x_{2n-1} = z_n$, $x_{2n} = w_n$ in the previous equations we get

$$z_{n+1} = A_0 + \frac{z_n^p}{w_n^q}, \;\; w_{n+1} = A_1 + \frac{w_n^p}{z_{n+1}^q}. \tag{3.50}$$

Then $(\bar{x}, \bar{y})$ is the positive solution of (3.50).
The system

$$z_{n+1} = A_0 + \frac{z_n^p}{w_n^q}, \;\; w_{n+1} = A_1 + \frac{w_n^p}{z_{n+1}^q}$$

can be written as

$$z_{n+1} = A_0 + \frac{z_n^p}{w_n^q}, \;\; w_{n+1} = A_1 + \frac{w_n^p}{\left(A_0 + \frac{z_n^p}{w_n^q}\right)^q},$$

which can be linearized as

$$\nu_{n+1} = B\nu_n,$$

where

$$B = \begin{pmatrix} \frac{\partial z_{n+1}}{\partial z_n} & \frac{\partial z_{n+1}}{\partial w_n} \\ \frac{\partial w_{n+1}}{\partial z_n} & \frac{\partial w_{n+1}}{\partial w_n} \end{pmatrix}, \; \nu_n = \begin{pmatrix} z_n \\ w_n \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{w_n p z_n^{p-1} - 0}{w_n^{2q}} & \frac{0 - z_n^p q w_n^{q-1}}{w_n^{2q}} \\ \frac{0 - w_n^p q \left(A_0 + \frac{z_n^p}{w_n^q}\right)^{q-1} \left(\frac{w_n^q p z_n^{p-1} - 0}{w_n^{2q}}\right)}{\left(A_0 + \frac{z_n^p}{w_n^q}\right)^{2q}} & \frac{\left(A_0 + \frac{z_n^p}{w_n^q}\right)^q p w_n^{p-1} - w_n^p q \left(A_0 + \frac{z_n^p}{w_n^q}\right)^{q-1} \left(\frac{0 - z_n^p q w_n^{q-1}}{w_n^{2q}}\right)}{\left(A_0 + \frac{z_n^p}{w_n^q}\right)^{2q}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{p}{z_n^{1-p} w_n^q} & \frac{-q z_n^p}{w_n^{q+1}} \\ \frac{-pq}{\left(A_0 + \frac{z_n^p}{w_n^q}\right)^{q+1} w_n^{q-p} z_n^{1-p}} & \frac{p}{w_n^{1-p}\left(A_0 + \frac{z_n^p}{w_n^q}\right)^q} + \frac{q^2}{w_{q+1-p} z_n^{-p} \left(A_0 + \frac{z_n^p}{w_n^q}\right)^{q+1}} \end{pmatrix}.$$

When this system is evaluated at $(\bar{x}, \bar{y})$ we get

$$B = \begin{pmatrix} \frac{p}{\bar{x}^{1-p}\bar{y}^q} & \frac{-q\bar{x}^p}{\bar{y}^{q+1}} \\ \frac{-qp}{\bar{y}^{q-p}\bar{x}^{1-p}\left(A_0 + \frac{\bar{x}^p}{\bar{y}^q}\right)^{q+1}} & \frac{p}{\bar{y}^{1-p}\left(A_0 + \frac{\bar{x}^p}{\bar{y}^q}\right)^q} + \frac{q^2}{\bar{y}^{1-p+q}\bar{x}^{-p}\left(A_0 + \frac{\bar{x}^p}{\bar{y}^q}\right)^{q+1}} \end{pmatrix}.$$

The solution is two periodic, so $\bar{x} = x_{2n-1} = x_{2n+1} = A_0 + \frac{x_{2n-1}^p}{x_{2n}^q} = A_0 + \frac{\bar{x}^p}{\bar{y}^q}$. Then

$$B = \begin{pmatrix} \frac{p}{\bar{x}^{1-p}\bar{y}^q} & \frac{-q\bar{x}^p}{\bar{y}^{q+1}} \\ \frac{-pq}{\bar{x}^{q+2-p}\bar{y}^{q-p}} & \frac{p}{\bar{x}^q\bar{y}^{1-p}} + \frac{q^2}{(\bar{x}\bar{y})^{q+1-p}} \end{pmatrix}.$$

To get the characteristic equation of B we solve $|B - \lambda I| = 0$.

$$\begin{vmatrix} \frac{p}{\bar{x}^{1-p}\bar{y}^q} - \lambda & \frac{-q\bar{x}^p}{\bar{y}^{q+1}} \\ \frac{-pq}{\bar{x}^{q+2-p}\bar{y}^{q-p}} & \frac{p}{\bar{x}^q\bar{y}^{1-p}} + \frac{q^2}{(\bar{x}\bar{y})^{q+1-p}} - \lambda \end{vmatrix} = 0.$$

We get

$$\left(\frac{p}{\bar{x}^{1-p}\bar{y}^q} - \lambda\right)\left(\frac{p}{\bar{x}^q\bar{y}^{1-p}} + \frac{q^2}{(\bar{x}\bar{y})^{q+1-p}} - \lambda\right) - \left(\frac{-q\bar{x}^p}{\bar{y}^{q+1}}\right)\left(\frac{-pq}{\bar{x}^{q+2-p}\bar{y}^{q-p}}\right) = 0.$$

Then

$$\frac{p^2}{\bar{x}^{q-p+1}\bar{y}^{q-p+1}}+\frac{pq^2}{\bar{x}^{q-2p+2}\bar{y}^{2q-p+1}}-\frac{p\lambda}{\bar{x}^{1-p}\bar{y}^{q}}-\frac{p\lambda}{\bar{x}^{q}\bar{y}^{1-p}}-\frac{q^2\lambda}{(\bar{x}\bar{y})^{q+1-p}}+\lambda^2-\frac{pq^2}{\bar{x}^{q-2p+2}\bar{y}^{2q-p+1}}=0.$$

The characteristic equation of B is the following

$$\lambda^2-\lambda\left(\frac{p}{\bar{x}^{1-p}\bar{y}^{q}}+\frac{p}{\bar{x}^{q}\bar{y}^{1-p}}+\frac{q^2}{(\bar{x}\bar{y})^{q+1-p}}\right)+\frac{p^2}{(\bar{x}\bar{y})^{q+1-p}}=0. \tag{3.51}$$

Since the solution is two periodic then $\bar{x}$, $\bar{y}$ satisfy (3.31) and we have that $\bar{x}>A_0$, $\bar{y}>A_1$ and so (3.47) implies

$$\begin{aligned}
&\frac{p}{\bar{x}^{1-p}\bar{y}^{q}}+\frac{p}{\bar{x}^{q}\bar{y}^{1-p}}+\frac{q^2}{(\bar{x}\bar{y})^{q+1-p}}\\
&<\frac{p}{\bar{x}^{1-p}\bar{y}^{q}}+\frac{p}{\bar{x}^{q}\bar{y}^{1-p}}+\frac{p^2+q^2}{(\bar{x}\bar{y})^{q+1-p}}\\
&<\frac{p}{A_1^qA_0^{1-p}}+\frac{p}{A_0^qA_1^{1-p}}+\frac{p^2+q^2}{(A_1A_0)^{q+1-p}}<1.
\end{aligned}$$

So all the roots of (3.51) are of modulus less than 1. Hence, using Theorem 1.3.1 $(\bar{x},\bar{y})$ is locally asymptotically stable, and referring to proposition 3.2.1 the solution is globally asymptotically stable.

(ii) We conclude from proposition 3.2.1 that there exists a unique 3-periodic solution $\bar{x}_n$ of (3.1).

Let

$$x_{3n-1}=\bar{x},\ x_{3n}=\bar{y},\ x_{3n+1}=A_0+\frac{x_{3n-1}^p}{x_{3n}^q}=A_0+\frac{\bar{x}^p}{\bar{y}^q}=\bar{z},\ n=0,1,....$$

From (3.1) we get

$$x_{3n+1}=A_0+\frac{x_{3n-1}^p}{x_{3n}^q},\ x_{3n+2}=A_1+\frac{x_{3n}^p}{x_{3n+1}^q},\ x_{3n+3}=A_2+\frac{x_{3n+1}^p}{x_{3n+2}^q},\ n=0,1,.... \tag{3.52}$$

If we set $x_{3n-2}=u_n$, $x_{3n-1}=\nu_n$, $x_{3n}=w_n$ in (3.52) we get

$$u_{n+1}=A_0+\frac{\nu_n^p}{w_n^q},\ \nu_{n+1}=A_1+\frac{w_n^p}{u_{n+1}^q},\ w_{n+1}=A_2+\frac{u_{n+1}^p}{\nu_{n+1}^q},\ n=0,1,.... \tag{3.53}$$

Then $(\bar{z}, \bar{x}, \bar{y})$ is the positive equilibrium point of (3.53). The preceding system can be written as

$$u_{n+1} = A_0 + \frac{\nu_n^p}{w_n^q},\ \nu_{n+1} = A_1 + \frac{w_n^p}{\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^q},\ w_{n+1} = A_2 + \frac{\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^p}{\left(A_1 + \frac{w_n^p}{\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^q}\right)^q},$$

which can be linearized as

$$z_{n+1} = T z_n,$$

where

$$T = \begin{pmatrix} \frac{\partial u_{n+1}}{\partial u_n} & \frac{\partial u_{n+1}}{\partial \nu_n} & \frac{\partial u_{n+1}}{\partial w_n} \\ \frac{\partial \nu_{n+1}}{\partial u_n} & \frac{\partial \nu_{n+1}}{\partial \nu_n} & \frac{\partial \nu_{n+1}}{\partial w_n} \\ \frac{\partial w_{n+1}}{\partial u_n} & \frac{\partial w_{n+1}}{\partial \nu_n} & \frac{\partial w_{n+1}}{\partial w_n} \end{pmatrix} = \begin{pmatrix} 0 & r_1 & s_1 \\ 0 & r_2 & s_2 \\ 0 & r_3 & s_3 \end{pmatrix},\ z_n = \begin{pmatrix} u_n \\ \nu_n \\ w_n \end{pmatrix}.$$

Now,

$$\frac{\partial \left(A_0 + \frac{\nu_n^p}{w_n^q}\right)}{\partial u_n} = 0.$$

$$r_1 = \frac{\partial \left(A_0 + \frac{\nu_n^p}{w_n^q}\right)}{\partial \nu_n} = \frac{w_n^q p \nu_n^{p-1} - 0}{w_n^{2q}} = \frac{p}{\nu_n^{1-p} w_n^q}.$$

$$s_1 = \frac{\partial \left(A_0 + \frac{\nu_n^p}{w_n^q}\right)}{\partial w_n} = \frac{0 - \nu_n^p q w_n^{q-1}}{w_n^{2q}} = \frac{-q\nu_n^p}{w_n^{q+1}}.$$

$$\frac{\partial \left(A_1 + \frac{w_n^p}{\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^q}\right)}{\partial u_n} = 0.$$

$$r_2 = \frac{\partial \left(A_1 + \frac{w_n^p}{\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^q}\right)}{\partial \nu_n}$$

$$= \frac{0 - w_n^p q \left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^{q-1} \left(\frac{p w_n^q \nu_n^{p-1} - 0}{w_n^{2q}}\right)}{\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^{2q}}$$

$$= \frac{-w_n^p q \left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^{q-1} \left(\frac{p w_n^q \nu_n^{p-1}}{w_n^{2q}}\right)}{\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^{2q}}$$

$$r_2 = \frac{-pq}{w_n^{q-p}\nu_n^{1-p}\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^{q+1}}$$

$$s_2 = \frac{\partial\left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q}\right)}{\partial w_n}$$

$$= \frac{\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^q p w_n^{p-1} - w_n^p q\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^{q-1}\left(\frac{0-\nu_n^p q w_n^{q-1}}{w_n^{2q}}\right)}{\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^{2q}}$$

$$= \frac{p}{w_n^{1-p}\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^q} + \frac{q^2\nu_n^p}{\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^{q+1} w_n^{q+1-p}}.$$

$$\frac{\partial\left(A_2 + \frac{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^p}{\left(A_1+\frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q}\right)^q}\right)}{\partial u_n} = 0.$$

$$r_3 = \frac{\partial\left(A_2 + \frac{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^p}{\left(A_1+\frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q}\right)^q}\right)}{\partial \nu_n}$$

$$= \frac{\left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q}\right)^q p\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^{p-1}\left[\frac{w_n^q p\nu_n^{p-1}-0}{w_n^{2q}}\right]}{\left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q}\right)^{2q}}$$

$$-\frac{\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^p q\left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q}\right)^{q-1}\left[0 - w_n^p q\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^{q-1}\left(\frac{w_n^q p\nu_n^{p-1}-0}{w_n^{2q}}\right)\right]}{\left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q}\right)^{2q}\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^{2q}}.$$

$$r_3 = \frac{p^2}{\nu^{1-p} w_n^q \left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q} \right)^q \left(A_0 + \frac{\nu_n^p\,q}{w\ n} \right)^{1-p}}$$

$$+\frac{pq^2}{\nu_n^{1-p} w_n^{q-p} \left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q} \right)^{q+1} \left(A_0 + \frac{\nu_n^p}{w_n^q} \right)^{q+1-p}}.$$

$$s_3 = \frac{\partial \left(A_2 + \frac{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^p}{\left(A_1+\frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q}\right)^q} \right)}{\partial w_n}.$$

$$s_3 = \frac{\left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q} \right)^q p \left(A_0 + \frac{\nu_n^p}{w_n^q} \right)^{p-1} \left(\frac{0-\nu_n^p q w_n^{q-1}}{w_n^{2q}} \right)}{\left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q} \right)^{2q}}$$

$$-\frac{\left(A_0 + \frac{\nu_n^p}{w_n^q} \right)^p q \left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q} \right)^{q-1} \left[\left(A_0 + \frac{\nu_n^p}{w_n^q} \right)^q p w_n^{p-1} - w_n^p q \left(A_0 + \frac{\nu_n^p}{w_n^q} \right)^{q-1} \left(\frac{0-\nu_n^p q w_n^{q-1}}{w_n^{2q}} \right) \right]}{\left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q} \right)^{2q} \left(A_0 + \frac{\nu_n^p}{w_n^q} \right)^{2q}}.$$

$$s_3 = \frac{-pq\nu_n^p}{w_n^{q+1} \left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q} \right)^q \left(A_0 + \frac{\nu_n^p}{w_n^q} \right)^{1-p}} - \frac{qp}{w_n^{1-p} \left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q} \right)^{q+1} \left(A_0 + \frac{\nu_n^p}{w_n^q} \right)^{q-p}}$$

$$-\frac{q^3\nu_n^p}{w_n^{q+1-p} \left(A_0 + \frac{\nu_n^p}{w_n^q} \right)^{q+1-p} \left(A_1 + \frac{w_n^p}{\left(A_0+\frac{\nu_n^p}{w_n^q}\right)^q} \right)^{q+1}}.$$

Now, $u_n = x_{3n-2} = x_{3n+1} = \bar{z}$ since the solution x_n is three periodic, $\nu_n = x_{3n-1} = \bar{x}$, $w_n = x_{3n} = \bar{y}$. It is known that $A_0 + \frac{\nu_n^p}{w_n^q} = u_{n+1} = x_{3n+1} = x_{3n-2} = \bar{z}$, also we have that $A_1 + \frac{w_n^p}{\left(A_0 + \frac{\nu_n^p}{w_n^q}\right)^q} = \nu_{n+1} = x_{3n+2} = x_{3n-1} = \nu_n = \bar{x}$.

Consequently,

$$r_1 = \frac{p}{\bar{x}^{1-p}\bar{y}^q},$$

$$s_1 = -\frac{q\bar{x}^p}{\bar{y}^{q+1}},$$

$$r_2 = -\frac{pq}{\bar{x}^{1-p}\bar{y}^{q-p}\bar{z}^{q+1}},$$

$$s_2 = \frac{p}{\bar{y}^{1-p}\bar{z}^q} + \frac{q^2\bar{x}^p}{\bar{y}^{q+1-p}\bar{z}^{q+1}}$$

$$r_3 = \frac{p^2}{\bar{x}^{q+1-p}\bar{y}^q\bar{z}^{1-p}} + \frac{pq^2}{\bar{x}^{2+q-p}\bar{y}^{q-p}\bar{z}^{q+1-p}},$$

$$s_3 = -\frac{pq}{\bar{x}^{q-p}\bar{y}^{q+1}\bar{z}^{1-p}} - \frac{pq}{\bar{x}^{1+q}\bar{y}^{1-p}\bar{z}^{q-p}} - \frac{q^3}{(\bar{x}\bar{y}\bar{z})^{q+1-p}}.$$

Now, $|T - \lambda I| = 0$

$$|T - \lambda I| = \begin{vmatrix} -\lambda & r_1 & s_1 \\ 0 & r_2 - \lambda & s_2 \\ 0 & r_3 & s_3 - \lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & \frac{p}{\bar{x}^{1-p}\bar{y}^q} & -\frac{q\bar{x}^p}{\bar{y}^{q+1}} \\ 0 & -\frac{pq}{\bar{x}^{1-p}\bar{y}^{q-p}\bar{z}^{q+1}} - \lambda & \frac{p}{\bar{y}^{1-p}\bar{z}^q} + \frac{q^2\bar{x}^p}{\bar{y}^{q+1-p}\bar{z}^{q+1}} \\ 0 & \frac{p^2}{\bar{x}^{q+1-p}\bar{y}^q\bar{z}^{1-p}} + \frac{pq^2}{\bar{x}^{2+q-p}\bar{y}^{q-p}\bar{z}^{q+1-p}} & -\frac{pq}{\bar{x}^{q-p}\bar{y}^{q+1}\bar{z}^{1-p}} - \frac{pq}{\bar{x}^{1+q}\bar{y}^{1-p}\bar{z}^{q-p}} - \frac{q^3}{(\bar{x}\bar{y}\bar{z})^{q+1-p}} - \lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\frac{pq}{\bar{x}^{1-p}\bar{y}^{q-p}\bar{z}^{q+1}} - \lambda & \frac{p}{\bar{y}^{1-p}\bar{z}^q} + \frac{q^2\bar{x}^p}{\bar{y}^{q+1-p}\bar{z}^{q+1}} \\ \frac{p^2}{\bar{x}^{q+1-p}\bar{y}^q\bar{z}^{1-p}} + \frac{pq^2}{\bar{x}^{2+q-p}\bar{y}^{q-p}\bar{z}^{q+1-p}} & -\frac{pq}{\bar{x}^{q-p}\bar{y}^{q+1}\bar{z}^{1-p}} - \frac{pq}{\bar{x}^{1+q}\bar{y}^{1-p}\bar{z}^{q-p}} - \frac{q^3}{(\bar{x}\bar{y}\bar{z})^{q+1-p}} - \lambda \end{vmatrix} = 0,$$

which is

$$-\lambda \left(\left(-\frac{pq}{\bar{x}^{1-p}\bar{y}^{q-p}\bar{z}^{q+1}} - \lambda \right) \left(-\frac{pq}{\bar{x}^{q-p}\bar{y}^{q+1}\bar{z}^{1-p}} - \frac{pq}{\bar{x}^{1+q}\bar{y}^{1-p}\bar{z}^{q-p}} - \frac{q^3}{(\bar{x}\bar{y}\bar{z})^{q+1-p}} - \lambda \right) \right.$$
$$\left. - \left(\frac{p}{\bar{y}^{1-p}\bar{z}^q} + \frac{q^2\bar{x}^p}{\bar{y}^{q+1-p}\bar{z}^{q+1}} \right) \left(\frac{p^2}{\bar{x}^{q-p+1}\bar{y}^q\bar{z}^{1-p}} + \frac{pq^2}{\bar{x}^{2+q-p}\bar{y}^{q-p}\bar{z}^{q+1-p}} \right) \right) = 0$$

Then

$$-\lambda\left(\frac{(pq)^2}{\bar{x}^{q-2p+1}\bar{y}^{2q-p+1}\bar{z}^{q-p+2}}+\frac{(pq)^2}{\bar{x}^{q-p+2}\bar{y}^{q-2p+1}\bar{z}^{2q-p+1}}+\frac{pq^4}{\bar{x}^{q-2p+2}\bar{y}^{2q-2p+1}\bar{z}^{2q-p+2}}\right.$$
$$+\frac{\lambda pq}{\bar{x}^{1-p}\bar{y}^{q-p}\bar{z}^{q+1}}+\frac{\lambda pq}{\bar{x}^{q-p}\bar{y}^{q+1}\bar{z}^{1-p}}+\frac{\lambda pq}{\bar{x}^{q+1}\bar{y}^{1-p}\bar{z}^{q-p}}+\frac{\lambda q^3}{(\bar{x}\bar{y}\bar{z})^{q+1-p}}+\lambda^2$$
$$\left.-\frac{p^3}{(\bar{x}\bar{y}\bar{z})^{q-p+1}}-\frac{(pq)^2}{\bar{x}^{q-p+2}\bar{y}^{q-2p+1}\bar{z}^{2q-p+1}}-\frac{(pq)^2}{\bar{x}^{q-2p+1}\bar{y}^{2q-p+1}\bar{z}^{q-p+2}}-\frac{pq^4}{\bar{x}^{q-2p+2}\bar{y}^{2q-2p+1}\bar{z}^{2q-p+2}}\right)=0,$$

which implies that

$$\lambda\left(\lambda^2+\frac{\lambda pq}{\bar{x}^{1-p}\bar{y}^{q-p}\bar{z}^{q+1}}+\frac{\lambda pq}{\bar{x}^{q-p}\bar{y}^{q+1}\bar{z}^{1-p}}+\frac{\lambda pq}{\bar{x}^{q+1}\bar{y}^{1-p}\bar{z}^{q-p}}+\frac{\lambda q^3}{(\bar{x}\bar{y}\bar{z})^{q-p+1}}-\frac{p^3}{(\bar{x}\bar{y}\bar{z})^{q-p+1}}\right)=0.$$

Then the characteristic equation of the matrix T is

$$\lambda\left(\lambda^2+\lambda\left(\frac{pq}{\bar{x}^{q-p}\bar{y}^{q+1}\bar{z}^{1-p}}+\frac{pq}{\bar{x}^{1-p}\bar{y}^{q-p}\bar{z}^{q+1}}+\frac{pq}{\bar{x}^{1+q}\bar{y}^{1-p}\bar{z}^{q-p}}+\frac{q^3}{(\bar{x}\bar{y}\bar{z})^{q+1-p}}\right)-\frac{p^3}{(\bar{x}\bar{y}\bar{z})^{q+1-p}}\right)=0. \tag{3.54}$$

Now,

$$\frac{pq}{\bar{x}^{q-p}\bar{y}^{q+1}\bar{z}^{1-p}}+\frac{pq}{\bar{x}^{1-p}\bar{y}^{q-p}\bar{z}^{q+1}}+\frac{pq}{\bar{x}^{1+q}\bar{y}^{1-p}\bar{z}^{q-p}}+\frac{q^3}{(\bar{x}\bar{y}\bar{z})^{q+1-p}}<$$
$$\frac{pq}{A_0^{q-p}A_1^{q+1}A_2^{1-p}}+\frac{pq}{A_0^{1-p}A_1^{q-p}A_2^{q+1}}+\frac{pq}{A_0^{1+q}A_1^{1-p}A_2^{q-p}}+\frac{q^3}{(A_0A_1A_2)^{q+1-p}}<$$
$$\frac{pq}{A_0^{q-p}A_1^{q+1}A_2^{1-p}}+\frac{pq}{A_0^{1-p}A_1^{q-p}A_2^{q+1}}+\frac{pq}{A_0^{1+q}A_1^{1-p}A_2^{q-p}}+\frac{p^3+q^3}{(A_0A_1A_2)^{q+1-p}}$$
$$<\frac{3pq}{C^{2(q+p-1)}}+\frac{p^3+q^3}{C^{3(q+p-1)}}<1.$$

So all the roots of (3.54) are of modulus less than 1. Hence, according to Theorem 1.3.1 the unique 3-periodic solution of (3.1) is locally asymptotically stable.

Finally, from proposition (3.2.1) the unique 3-periodic solution is globally asymptotically stable. ■

Chapter 4

On the Difference Equation $x_{n+1} = p_n + \frac{x_n}{x_{n-1}}$

In this part we will study properties such as boundedness and persistence and attractivity of the equation

$$x_{n+1} = p_n + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, ... \tag{4.1}$$

Where $x_{-1} > 0$, $x_0 \geq 0$, and p_n is a positive bounded sequence with

$$\liminf_{n\to\infty} p_n = p \geq 0 \ \text{ and } \ \limsup_{n\to\infty} p_n = q < \infty. \tag{4.2}$$

4.1 Boundedness and persistence

Lemma 4.1.1 *Assume Eq.(4.2) is satisfied. Let x_n be a solution of (4.1)*

(i) If $p > 0$, then $\{x_n\}$ persists.

(ii) If $p > 1$, then $\{x_n\}$ is bounded from above.

Proof. (i) Assume that $p > 0$, it is clear that $x_n > 0$ for all $n = 1, 2, ...$, so $\frac{x_n}{x_{n-1}} > 0$, which concludes

$$x_{n+1} = p_n + \frac{x_n}{x_{n-1}} > p_n.$$

So we get

$$\liminf_{n\to\infty} x_n \geq \liminf_{n\to\infty} p_n = p.$$

Then

$$\liminf_{n\to\infty} x_n \geq p.$$

Thus $\{x_n\}$ persists.

(ii)Assume that $p > 1$, from (i) we know that $x_{n-1} \geq p_{n-2} \geq p - \epsilon > 1$ for sufficiently large n and $\epsilon > 0$. Use Eq.(4.1) to get

$$x_{n+1} = p_n + \frac{x_n}{x_{n-1}} \leq p_n + \frac{x_n}{p - \epsilon}.$$

Referring to Theorem (2.0.4), $\{x_n\}$ is bounded since p_n is bounded. ■

Lemma 4.1.2 *Assume that Eq.(4.2) is satisfied and* $p > 1$, *and let* x_n *be a solution of Eq.(4.1). If*

$$\lambda = \liminf_{n\to\infty} x_n \ \ \textit{and} \ \ \mu = \limsup_{n\to\infty} x_n,$$

then

$$\frac{pq-1}{q-1} \leq \lambda \leq \mu \leq \frac{pq-1}{p-1}. \tag{4.3}$$

Proof. Let $\epsilon > 0$, then there exists $N_0(\epsilon)$ such that for $n \geq N_0(\epsilon)$, we have $\lambda - \epsilon \leq x_n \leq \mu + \epsilon$ and $p - \epsilon \leq p_n \leq q + \epsilon$. Then,

$$x_{n+1} = p_n + \frac{x_n}{x_{n-1}} \geq p - \epsilon + \frac{\lambda - \epsilon}{\mu + \epsilon}, \tag{4.4}$$

and

$$x_{n+1} = p_n + \frac{x_n}{x_{n-1}} \leq q + \epsilon + \frac{\mu + \epsilon}{\lambda - \epsilon}. \tag{4.5}$$

As $n \to \infty$ we have

$$\lambda \geq p - \epsilon + \frac{\lambda - \epsilon}{\mu + \epsilon}, \tag{4.6}$$

and

$$\mu \leq q + \epsilon + \frac{\mu + \epsilon}{\lambda - \epsilon}. \tag{4.7}$$

$\epsilon > 0$ is arbitrary, so

$$\lambda \geq p + \frac{\lambda}{\mu},$$

and

$$\mu \le q + \frac{\mu}{\lambda}.$$

Hence,

$$\lambda\mu - p\mu \ge \lambda,$$

and

$$\lambda\mu - q\lambda \le \mu.$$

Hence,

$$\mu p + \lambda \le \lambda\mu \le q\lambda + \mu.$$

As a result, we get

$$\mu p - \mu \le q\lambda - \lambda,$$

and so

$$\mu(p-1) \le \lambda(q-1),$$

so we get

$$\frac{\mu}{\lambda} \le \frac{q-1}{p-1} \text{ and } \frac{\lambda}{\mu} \ge \frac{p-1}{q-1}.$$

For $n > N_0$ Eq. Using (4.4) and Eq.(4.5) and Taylor's expansion we get

$$\begin{aligned} x_{n+1} &\ge p - \epsilon + \frac{\lambda-\epsilon}{\mu+\epsilon} \\ &= p + \frac{\lambda}{\mu} + O(\epsilon) \\ &\ge p + \frac{p-1}{q-1} + O(\epsilon) \\ &= \frac{pq - p + p - 1}{q-1} + O(\epsilon) \\ &= \frac{pq-1}{q-1} + O(\epsilon), \end{aligned}$$

and

$$\begin{aligned} x_{n+1} &\leq q + \epsilon + \frac{\mu + \epsilon}{\lambda - \epsilon} \\ &= q + \frac{\mu}{\lambda} + O(\epsilon) \\ &\leq q + \frac{q-1}{p-1} + O(\epsilon) \\ &= \frac{pq - q + q - 1}{p-1} + O(\epsilon) \\ &= \frac{pq-1}{p-1} + O(\epsilon). \end{aligned}$$

Now, $\epsilon > 0$ is arbitrary, then,

$$x_{n+1} \geq \frac{pq-1}{q-1},$$

and

$$x_{n+1} \leq \frac{pq-1}{p-1}.$$

As $n \to \infty$, we have

$$\lambda \geq \frac{pq-1}{q-1},$$

and

$$\mu \leq \frac{pq-1}{p-1}.$$

So we get

$$\frac{pq-1}{q-1} \leq \lambda \leq \mu \leq \frac{pq-1}{p-1}.$$

■

Theorem 4.1.1 *Consider the interval* $I = [\frac{PQ-1}{Q-1}, \frac{PQ-1}{P-1}]$*, where* $1 < P \leq p_n \leq Q$*, for* $n = 0, 1, 2, ...$ *If* x_n *is a solution of Eq.(4.1) such that* $x_{-1}, x_0 \in I$*, then* $x_n \in I$ *for all* $n = 0, 1, 2, ...$

Proof.

$$x_1 = p_0 + \frac{x_0}{x_{-1}}.$$

Now, $x_{-1}, x_0 \in I = [\frac{PQ-1}{Q-1}, \frac{PQ-1}{P-1}]$, so we get

$$\frac{x_0}{x_{-1}} \leq \frac{\frac{PQ-1}{P-1}}{\frac{PQ-1}{Q-1}} = \frac{Q-1}{P-1},$$

and
$$\frac{x_0}{x_{-1}} \geq \frac{\frac{PQ-1}{Q-1}}{\frac{PQ-1}{P-1}} = \frac{P-1}{Q-1}.$$

Then,

$$x_1 = p_0 + \frac{x_0}{x_{-1}} \leq Q + \frac{\frac{PQ-1}{P-1}}{\frac{PQ-1}{Q-1}} = Q + \frac{Q-1}{P-1} = \frac{PQ-Q+Q-1}{P-1} = \frac{PQ-1}{P-1},$$

and

$$x_1 = p_0 + \frac{x_0}{x_{-1}} \geq P + \frac{\frac{PQ-1}{Q-1}}{\frac{PQ-1}{P-1}} = P + \frac{P-1}{Q-1} = \frac{PQ-P+P-1}{Q-1} = \frac{PQ-1}{Q-1}.$$

So $x_1 \in I$. Assume that the result holds for $k = 2, 3, ..., n$.
For $k = n+1$

$$x_{n+1} = p_n + \frac{x_n}{x_{n-1}} \leq Q + \frac{\frac{PQ-1}{p-1}}{\frac{PQ-1}{Q-1}} = Q + \frac{Q-1}{P-1} = \frac{PQ-Q+Q-1}{P-1} = \frac{PQ-1}{P-1},$$

and

$$x_{n+1} = p_n + \frac{x_n}{x_{n-1}} \geq P + \frac{\frac{PQ-1}{Q-1}}{\frac{PQ-1}{P-1}} = P + \frac{P-1}{Q-1} = \frac{PQ-P+P-1}{Q-1} = \frac{PQ-1}{Q-1}.$$

So $x_{n+1} \in I$. We conclude that $x_n \in I$, for all $n = 0, 1, ...$ ∎

4.2 Attractivity

Assume that $\bar{x}$ is a positive solution of (4.1). Here we are interested in finding sufficient conditions such that $\bar{x}$ attracts all the positive solutions of the equation, in other words we mean

$$x_n \to \bar{x}.$$

Now, let
$$y_n = \frac{x_n}{\bar{x}_n}, \quad n = -1, 0, 1, ...$$

This gives
$$x_n = \bar{x}_n y_n.$$

Returning to Eq.(4.1), plug in the new value of x_n

$$\bar{x}_{n+1}y_{n+1} = p_n + \frac{\bar{x}_n y_n}{\bar{x}_{n-1}y_{n-1}}.$$

$$\begin{aligned} y_{n+1} &= \frac{p_n + \frac{\bar{x}_n y_n}{\bar{x}_{n-1}y_{n-1}}}{\bar{x}_{n+1}} \\ &= \frac{p_n + \frac{\bar{x}_n y_n}{\bar{x}_{n-1}y_{n-1}}}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}}. \end{aligned}$$

Then

$$y_{n+1} = \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}\frac{y_n}{y_{n-1}}}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}}. \tag{4.8}$$

Lemma 4.2.1 *Let x_n be a positive solution of Eq.(4.1). Then the following are true*

(i) Eq.(4.8) has a positive equilibrium solution $\bar{y} = 1$.

(ii) If for some n, $y_{n-1} < y_n$, then $y_{n+1} > 1$. Similarly, if for some n, $y_{n-1} \geq y_n$, then $y_{n+1} \leq 1$.

Proof. (i)

$$\bar{y} = \frac{p_n + \frac{\bar{x}}{\bar{x}}\frac{y}{y}}{p_n + \frac{\bar{x}}{\bar{x}}} = 1.$$

Then Eq.(4.8) has a positive equilibrium solution that is 1.
(ii)Assume that for some n, $y_{n-1} < y_n$, then $\frac{y_n}{y_{n-1}} > 1$.

$$\begin{aligned} y_{n+1} &= \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}\left(\frac{y_n}{y_{n-1}}\right)}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} \\ &> \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}.(1)}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} \\ &= \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} = 1. \end{aligned}$$

So $y_{n+1} > 1$.
Similarly, assume $y_{n-1} \geq y_n$, then $\frac{y_n}{y_{n-1}} \leq 1$. Now,

$$
\begin{aligned}
y_{n+1} &= \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}\left(\frac{y_n}{y_{n-1}}\right)}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} \\
&\leq \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}.(1)}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} \\
&= \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} = 1.
\end{aligned}
$$

So $y_{n+1} \leq 1$. ■

Theorem 4.2.1 *Let y_n be a solution of Eq.(4.1)*

a) Assume that there exists n such that $y_{n-1} < 1$ and $y_n > 1$ and $y_{n+2} < y_{n+3} < y_{n+4} < ....$

(i) If $y_n > y_{n+1}$, then $y_{n+k} > 1$ for all $k = 4, 5, ...$

(ii) If $y_n < y_{n+1}$ and $y_{n+2} > y_{n+1}$, then $y_{n+k} > 1$ for all $k = 1, 2, ...$

b) Assume that there exists n such that $y_{n-1} > 1$ and $y_n < 1$ and $y_{n+2} < y_{n+3} < y_{n+4} < ....$

(i) If $y_n > y_{n+1}$ and $y_{n+2} > y_{n+1}$, then $y_{n+k} > 1$ for all $k = 3, 4, ...$

(ii) If $y_n < y_{n+1}$, then $y_{n+k} > 1$ for all $k = 2, 3, ...$

Proof. a) Assume that $y_{n-1} < 1$ and $y_n > 1$, then $\frac{y_n}{y_{n-1}} > 1$, which concludes that

$$
\begin{aligned}
y_{n+1} &= \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}\left(\frac{y_n}{y_{n-1}}\right)}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} \\
&> \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}.(1)}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} \\
&= \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} = 1.
\end{aligned}
$$

Thus, $y_{n+1} > 1$.
(i) Now, since $y_n > y_{n+1}$ and $\frac{y_{n+1}}{y_n} < 1$ and we have

$$y_{n+2} = \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}\left(\frac{y_{n+1}}{y_n}\right)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} < \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}.(1)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} = \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} = 1.$$

Hence $y_{n+2} < 1$, it's clear that $y_{n+2} < y_{n+1}$, then $\frac{y_{n+2}}{y_{n+1}} < 1$, thus

$$y_{n+3} = \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}\left(\frac{y_{n+2}}{y_{n+1}}\right)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} < \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}.(1)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = 1.$$

Thus $y_{n+3} < 1$.
Assume that $y_{n+2} < y_{n+3} < y_{n+4} < \ldots$.
For $k = 4$

$$y_{n+4} = \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}\left(\frac{y_{n+3}}{y_{n+2}}\right)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} > \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}.(1)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = 1.$$

Thus $y_{n+4} > 1$.

$$\begin{aligned} y_{n+5} &= \frac{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}} \left(\frac{y_{n+4}}{y_{n+3}} \right)}{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}} \\ &> \frac{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}.(1)}{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}} \\ &= \frac{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}}{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}} = 1. \end{aligned}$$

Then $y_{n+5} > 1$.

$$\begin{aligned} y_{n+6} &= \frac{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}} \left(\frac{y_{n+5}}{y_{n+4}} \right)}{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}} \\ &> \frac{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}.(1)}{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}} \\ &= \frac{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}}{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}} = 1. \end{aligned}$$

Thus $y_{n+6} > 1$.
It is obvious that for $k = 4, 5, 6, ...$ we have $y_{n+k} > 1$ since $y_{n+k-1} > y_{n+k-2}$, then $\frac{y_{n+k-1}}{y_{n+k-2}} > 1$. As a result,

$$y_{n+k} = \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}} \left(\frac{y_{n+k-1}}{y_{n+k-2}} \right)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} > \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}.(1)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} = 1.$$

(ii) Now, since $y_n < y_{n+1}$, $\frac{y_{n+1}}{y_n} > 1$, thus we have

$$\begin{aligned} y_{n+2} &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_{n}} \left(\frac{y_{n+1}}{y_{n}} \right)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_{n}}} \\ &> \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_{n}}.(1)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_{n}}} \\ &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_{n}}}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_{n}}} = 1. \end{aligned}$$

Thus $y_{n+2} > 1$.
Let $k = 3$. It is assumed that $y_{n+2} > y_{n+1}$, so $\frac{y_{n+2}}{y_{n+1}} > 1$, thus

$$\begin{aligned} y_{n+3} &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}\left(\frac{y_{n+2}}{y_{n+1}}\right)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &> \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}.(1)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = 1. \end{aligned}$$

Thus $y_{n+3} > 1$.
It is assumed that $y_{n+2} < y_{n+3}$, then $\frac{y_{n+3}}{y_{n+2}} > 1$, which implies that

$$\begin{aligned} y_{n+4} &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}\left(\frac{y_{n+3}}{y_{n+2}}\right)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &> \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}.(1)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = 1. \end{aligned}$$

So $y_{n+4} > 1$.
In general for $k = 1, 2, 3, ...$ it is true that $y_{n+k} > 1$ since $y_{n+k-1} > y_{n+k-2}$ and $\frac{y_{n+k-1}}{y_{n+k-2}} > 1$. As a result,

$$y_{n+k} = \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}\left(\frac{y_{n+k-1}}{y_{n+k-2}}\right)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} > \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}.(1)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} = 1.$$

b) Assume that $y_{n-1} > 1$ and $y_n < 1$, then $\frac{y_n}{y_{n-1}} < 1$, thus we have

$$\begin{aligned} y_{n+1} &= \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}\left(\frac{y_n}{y_{n-1}}\right)}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} \\ &< \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}.(1)}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} \\ &= \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}}{p_{n+1} + \frac{\bar{x}_n}{\bar{x}_{n-1}}} = 1. \end{aligned}$$

Thus $y_{n+1} < 1$.
i) Since $y_n > y_{n+1}$, $\frac{y_{n+1}}{y_n} < 1$, then

$$\begin{aligned} y_{n+2} &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}\left(\frac{y_{n+1}}{y_n}\right)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} \\ &< \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}.(1)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} \\ &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} = 1. \end{aligned}$$

Then $y_{n+2} < 1$. Now, for $k = 3$, it is given that $y_{n+2} > y_{n+1}$ which gives $\frac{y_{n+2}}{y_{n+1}} > 1$, consequently

$$\begin{aligned} y_{n+3} &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}\left(\frac{y_{n+2}}{y_{n+1}}\right)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &> \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}.(1)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = 1. \end{aligned}$$

So $y_{n+3} > 1$. It is assumed that $y_{n+2} < y_{n+3}$, then $\frac{y_{n+3}}{y_{n+2}} > 1$, which implies that

$$\begin{aligned} y_{n+4} &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}\left(\frac{y_{n+3}}{y_{n+2}}\right)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &> \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}.(1)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = 1. \end{aligned}$$

So $y_{n+4} > 1$.
Generally, for $k = 3, 4, ...$ we have $y_{n+k} > 1$ since $y_{n+k-1} > y_{n+k-2}$ and $\frac{y_{n+k-1}}{y_{n+k-2}} > 1$. As a result,

$$y_{n+k} = \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}\left(\frac{y_{n+k-1}}{y_{n+k-2}}\right)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} > \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}.(1)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} = 1.$$

(ii)For $k=2$, $y_n < y_{n+1}$, consequently $\frac{y_{n+1}}{y_n} > 1$ and

$$\begin{aligned} y_{n+2} &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}\left(\frac{y_{n+1}}{y_n}\right)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} \\ &> \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}.(1)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} \\ &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} = 1. \end{aligned}$$

Thus $y_{n+2} > 1$, since $y_{n+2} > y_{n+1}$ we have

$$\begin{aligned} y_{n+3} &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}\left(\frac{y_{n+2}}{y_{n+1}}\right)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &> \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}.(1)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = 1. \end{aligned}$$

For $k=4$, $y_{n+3} > y_{n+2}$, then $\frac{y_{n+3}}{y_{n+2}} > 1$, which gives

$$\begin{aligned} y_{n+4} &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}\left(\frac{y_{n+3}}{y_{n+2}}\right)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &> \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}.(1)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = 1. \end{aligned}$$

Hence, $y_{n+k} > 1$ for all $k = 2, 3, ...$ since $y_{n+k-1} > y_{n+k-2}$ and $\frac{y_{n+k-1}}{y_{n+k-2}} > 1$. As a result,

$$y_{n+k} = \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}\left(\frac{y_{n+k-1}}{y_{n+k-2}}\right)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} > \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}.(1)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} = 1.$$

■

Theorem 4.2.2 *Let y_n be a solution of Eq.(4.1)*
a)Assume that there exists n such that $y_{n-1} < 1$, $y_n > 1$ and $y_{n+2} > y_{n+3} > y_{n+4} > \ldots$.
(i) If $y_n > y_{n+1}$, then $y_{n+k} < 1$ for all $k = 2, 3, \ldots$.
(ii) If $y_n < y_{n+1}$ and $y_{n+2} < y_{n+1}$, then $y_{n+k} < 1$ for all $k = 3, 4, \ldots$.
b) Assume that there exists n such that $y_{n-1} > 1$, $y_n < 1$ and $y_{n+2} > y_{n+3} > y_{n+4} > \ldots$.
(i) If $y_n < y_{n+1}$, then $y_{n+k} < 1$ for all $k = 4, 5, \ldots$.
(ii) If $y_n > y_{n+1}$ and $y_{n+2} < y_{n+1}$, then $y_{n+k} < 1$ for all $k = 1, 2, \ldots$.

Proof. a) Assume that there exists n such that $y_{n-1} < 1$, $y_n > 1$ and $y_{n+2} > y_{n+3} > y_{n+4} > \ldots$.
It is clear that $\frac{y_n}{y_{n-1}} > 1$, then

$$\begin{aligned} y_{n+1} &= \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}\left(\frac{y_n}{y_{n-1}}\right)}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} \\ &> \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}.(1)}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} \\ &= \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}}{p_{n+1} + \frac{\bar{x}_n}{\bar{x}_{n-1}}} = 1. \end{aligned}$$

So $y_{n+1} > 1$.
(i)For $k = 2$

$$y_{n+2} = \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}\left(\frac{y_{n+1}}{y_n}\right)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}}.$$

Since $y_n > y_{n+1}$ we have $\frac{y_{n+1}}{y_n} < 1$, then

$$\begin{aligned} y_{n+2} &< \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}.(1)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} \\ &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} = 1. \end{aligned}$$

Thus $y_{n+2} < 1$.
For $k = 3$, $y_{n+2} < y_{n+1}$, then $\frac{y_{n+2}}{y_{n+1}} < 1$ and as a result

$$\begin{aligned} y_{n+3} &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}\left(\frac{y_{n+2}}{y_{n+1}}\right)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &< \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}.(1)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = 1. \end{aligned}$$

So $y_{n+3} < 1$.
Since $y_{n+2} > y_{n+3}$ and $\frac{y_{n+3}}{y_{n+2}} < 1$ we have

$$\begin{aligned} y_{n+4} &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}\left(\frac{y_{n+3}}{y_{n+2}}\right)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &< \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}.(1)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = 1. \end{aligned}$$

So $y_{n+4} < 1$.
Since $y_{n+k-1} < y_{n+k-2}$ we get $y_{n+k} < 1$ for all $k = 2, 3, ...$
In other words, $\frac{y_{n+k-1}}{y_{n+k-2}} < 1$. As a result,

$$y_{n+k} = \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}\left(\frac{y_{n+k-1}}{y_{n+k-2}}\right)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} < \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}.(1)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} = 1.$$

(ii) $y_{n+1} > y_n$, then $\frac{y_{n+1}}{y_n} > 1$ and so

$$\begin{aligned} y_{n+2} &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_{n}}\left(\frac{y_{n+1}}{y_{n}}\right)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_{n}}} \\ &> \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_{n}}.(1)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_{n}}} \\ &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_{n}}}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_{n}}} = 1. \end{aligned}$$

So $y_{n+2} > 1$.
For $k = 3$, $y_{n+2} < y_{n+1}$, then $\frac{y_{n+2}}{y_{n+1}} < 1$ and as a result

$$\begin{aligned} y_{n+3} &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}\left(\frac{y_{n+2}}{y_{n+1}}\right)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &< \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}.(1)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = 1. \end{aligned}$$

So $y_{n+3} < 1$. Since $y_{n+2} > y_{n+3}$ we have

$$\begin{aligned} y_{n+4} &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}\left(\frac{y_{n+3}}{y_{n+2}}\right)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &< \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}.(1)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = 1. \end{aligned}$$

So $y_{n+4} < 1$.
Hence $y_{n+k} < 1$ for all $k = 3, 4, ...$ since $y_{n+k-1} < y_{n+k-2}$.
In other words, $\frac{y_{n+k-1}}{y_{n+k-2}} < 1$. As a result,

$$y_{n+k} = \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}\left(\frac{y_{n+k-1}}{y_{n+k-2}}\right)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} < \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}.(1)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} = 1.$$

b) Assume that there exists n such that $y_{n-1} > 1$, $y_n < 1$, then $\frac{y_n}{y_{n-1}} < 1$ and

$$\begin{aligned} y_{n+1} &= \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}\left(\frac{y_n}{y_{n-1}}\right)}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} \\ &< \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}.(1)}{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}} \\ &= \frac{p_n + \frac{\bar{x}_n}{\bar{x}_{n-1}}}{p_{n+1} + \frac{\bar{x}_n}{\bar{x}_{n-1}}} = 1. \end{aligned}$$

(i) $y_n < y_{n+1}$ and consequently $\frac{y_{n+1}}{y_n} > 1$, then

$$\begin{aligned} y_{n+2} &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}\left(\frac{y_{n+1}}{y_n}\right)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} \\ &> \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}.(1)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} \\ &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} = 1. \end{aligned}$$

Then $y_{n+2} > 1$. It is clear that $y_{n+1} < y_{n+2}$, therefore

$$\begin{aligned} y_{n+3} &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}\left(\frac{y_{n+2}}{y_{n+1}}\right)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &> \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}.(1)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = 1. \end{aligned}$$

For $k = 4$, using the assumption that says $y_{n+2} > y_{n+3}$ we get $\frac{y_{n+3}}{y_{n+2}} < 1$, then

$$\begin{aligned} y_{n+4} &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}\left(\frac{y_{n+3}}{y_{n+2}}\right)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &< \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}.(1)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = 1. \end{aligned}$$

Then, $y_{n+4} < 1$.
Hence, $y_{n+k} < 1$ for $k = 4, 5, 6, ...$ as a result of $y_{n+k-1} < y_{n+k-2}$.
In other words, $\frac{y_{n+k-1}}{y_{n+k-2}} < 1$. As a result,

$$y_{n+k} = \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}\left(\frac{y_{n+k-1}}{y_{n+k-2}}\right)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} < \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}.(1)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} = 1.$$

(ii)For $k = 2$, we are given that $y_n > y_{n+1}$ which gives that $\frac{y_{n+1}}{y_n} < 1$, then

$$\begin{aligned} y_{n+2} &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}\left(\frac{y_{n+1}}{y_n}\right)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} \\ &< \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}.(1)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} \\ &= \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} = 1. \end{aligned}$$

So $y_{n+2} < 1$. As a result of the assumption we have $\frac{y_{n+2}}{y_{n+1}} < 1$, hence

$$\begin{aligned} y_{n+3} &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}\left(\frac{y_{n+2}}{y_{n+1}}\right)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &< \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}.(1)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} \\ &= \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = 1. \end{aligned}$$

Hence, $y_{n+3} < 1$. Now, for $k = 4$ we have

$$\begin{aligned} y_{n+4} &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}\left(\frac{y_{n+3}}{y_{n+2}}\right)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &< \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}.(1)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} \\ &= \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = 1. \end{aligned}$$

Thus $y_{n+4} < 1$.
Hence, $y_{n+k} < 1$ for $k = 1, 2, ...$ since $y_{n+k-1} < y_{n+k-2}$.
In other words, $\frac{y_{n+k-1}}{y_{n+k-2}} < 1$. As a result,

$$y_{n+k} = \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}\left(\frac{y_{n+k-1}}{y_{n+k-2}}\right)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} < \frac{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}.(1)}{p_{n+k-1} + \frac{\bar{x}_{n+k-1}}{\bar{x}_{n+k-2}}} = 1.$$

■

Theorem 4.2.3 *Let $\{y_n\}$ be a solution of Eq.(4.1). If there exists n such that $y_n > y_{n+1} > 1$ and $\frac{y_{n+k}}{y_{n+k-1}} > 1$ for $k = 4l-1$, where $l = 1, 2, 3, ...,$ and $\frac{y_{n+k}}{y_{n+k-1}} < 1$ for $k = 4l+1$, where $l = 0, 1, 2, ...,$ then $\{y_n\}$ is an oscillatory solution in which $\{y_{n+4l}, y_{n+4l+1}\}$ where $l = 1, 2, 3, ...$ gives the positive semicycles and $\{y_{n+4l+2}, y_{n+4l+3}\}$ where $l = 0, 1, 2, ...$ gives the negative semicycles.*

Proof. We proceed by induction. We are given that $y_n > y_{n+1}$ then $\frac{y_{n+1}}{y_n} < 1$, which implies that

$$y_{n+2} = \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}\left(\frac{y_{n+1}}{y_n}\right)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} < \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}.(1)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} = 1.$$

Then $y_{n+2} < 1$.
It is clear that $y_{n+2} < y_{n+1}$ which gives $\frac{y_{n+2}}{y_{n+1}} < 1$, then

$$y_{n+3} = \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}\left(\frac{y_{n+2}}{y_{n+1}}\right)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} < \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}.(1)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = 1.$$

Then $y_{n+3} < 1$. Now,

$$y_{n+4} = \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}\left(\frac{y_{n+3}}{y_{n+2}}\right)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} > \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}.(1)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = 1.$$

Since according to the assumption for $k = 4 \times 1 - 1 = 3$ we have $\frac{y_{n+3}}{y_{n+2}} > 1$, then $y_{n+4} > 1$.

$$y_{n+5} = \frac{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}\left(\frac{y_{n+4}}{y_{n+3}}\right)}{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}} > \frac{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}.(1)}{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}} = 1.$$

And that because $y_{n+4} > 1$ and $y_{n+3} < 1$, then $\frac{y_{n+4}}{y_{n+3}} > 1$, then $y_{n+5} > 1$.

$$y_{n+6} = \frac{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}\left(\frac{y_{n+5}}{y_{n+4}}\right)}{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}} < \frac{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}.(1)}{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}} = 1.$$

Since for $k = 4 \times 1 + 1 = 5$ we have that $\frac{y_{n+5}}{y_{n+4}} < 1$ as in the assumption, then $y_{n+6} < 1$. And

$$y_{n+7} = \frac{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}\left(\frac{y_{n+6}}{y_{n+5}}\right)}{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}} < \frac{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}.(1)}{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}} = 1.$$

Since $y_{n+6} < y_{n+5}$ we have $\frac{y_{n+6}}{y_{n+5}} < 1$, then $y_{n+7} < 1$.
If $k = 4 \times 2 - 1 = 7$, then $\frac{y_{n+7}}{y_{n+6}} > 1$ and we get

$$y_{n+8} = \frac{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}\left(\frac{y_{n+7}}{y_{n+6}}\right)}{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}} > \frac{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}.(1)}{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}} = 1.$$

Now, $y_{n+8} > 1$ and $y_{n+7} < 1$, then $\frac{y_{n+8}}{y_{n+7}} > 1$, as a result

$$y_{n+9} = \frac{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+7}}\left(\frac{y_{n+8}}{y_{n+7}}\right)}{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+8}}} > \frac{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+7}}.(1)}{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+7}}} = 1.$$

If $k = 4 \times 2 + 1 = 9$, then $\frac{y_{n+9}}{y_{n+8}} < 1$, then

$$y_{n+10} = \frac{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}\left(\frac{y_{n+9}}{y_{n+8}}\right)}{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}} < \frac{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}.(1)}{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}} = 1.$$

Obviously, $y_{n+11} < 1$.
Then $\{y_n, y_{n+1}\}$ is a positive semicycle using the assumption $y_n > y_{n+1} > 1$, $\{y_{n+2}, y_{n+3}\}$ is a negative semicycle, $\{y_{n+4}, y_{n+5}\}$ is a positive semicycle, $\{y_{n+6}, y_{n+7}\}$ is a negative semicycle, $\{y_{n+8}, y_{n+9}\}$ is a positive semicycle, and $\{y_{n+10}, y_{n+11}\}$ is a negative semicycle. If you used $l = 0$ you will get the second semicycle, and $l = 1$ gives the second two semicycles, and $l = 2$ gives the third two semicycles.
Assume that the result holds for $l - 1$, we prove it for l.
If for $k = 4l - 1$, $l = 1, 2, 3, ...$, $\frac{y_{n+k}}{y_{n+k-1}} > 1$, and if for $k = 4l + 1$, $l = 0, 1, 2, ...$, $\frac{y_{n+k}}{y_{n+k-1}} < 1$, then $\{y_{n+4l}, y_{n+4l+1}\}$ is a positive semicycle and $\{y_{n+4l+2}, y_{n+4l+3}\}$ is a negative semicycle.

$$y_{n+k+1} = \frac{p_{n+k} + \frac{\bar{x}_{n+k}}{\bar{x}_{n+k-1}}\left(\frac{y_{n+k}}{y_{n+k-1}}\right)}{p_{n+k} + \frac{\bar{x}_{n+k}}{\bar{x}_{n+k-1}}}$$

$$y_{n+4l} = \frac{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}\left(\frac{y_{n+4l-1}}{y_{n+4l-2}}\right)}{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}} > \frac{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}.(1)}{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}} = 1.$$

Consequently, $y_{n+4l+1} > 1$ by induction hypothesis. Now, using the second assumption we have

$$y_{n+4l+2} = \frac{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}\left(\frac{y_{n+4l+1}}{y_{n+4l}}\right)}{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}} < \frac{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}.(1)}{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}} = 1.$$

It is clear that $y_{n+4l+3} < 1$. As a result, $\{y_{n+4l}, y_{n+4l+1}\}$ is a positive semicycle and $\{y_{n+4l+2}, y_{n+4l+3}\}$ is a negative semicycle. ∎

Theorem 4.2.4 *Let $\{y_n\}$ be a solution of Eq.(4.1). If there exists n such that $y_{n+1} > y_n > 1$ and $\frac{y_{n+k}}{y_{n+k-1}} < 1$ for $k = 4l - 1$, where $l = 1, 2, 3, ...$, and $\frac{y_{n+k}}{y_{n+k-1}} > 1$ for $k = 2$ and $k = 4l + 1$, where $l = 0, 1, 2, ...$, then $\{y_n\}$ is an oscillatory solution in which $\{y_{n+4l}, y_{n+4l+1}\}$ where $l = 1, 2, 3, ...$ gives all negative semicycles and $\{y_{n+4l+2}, y_{n+4l+3}\}$ where $l = 0, 1, 2, ...$ gives all positive semicycles except the first one.*

Proof. We proceed by induction. We are given that $y_{n+1} > y_n$ then $\frac{y_{n+1}}{y_n} > 1$, which implies that

$$y_{n+2} = \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}\left(\frac{y_{n+1}}{y_n}\right)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} > \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}.(1)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} = 1.$$

Then $y_{n+2} > 1$.
Using the assumption $\frac{y_{n+2}}{y_{n+1}} > 1$, this implies that

$$y_{n+3} = \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}\left(\frac{y_{n+2}}{y_{n+1}}\right)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} > \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}.(1)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = 1.$$

Then $y_{n+3} > 1$, and $\{y_{n+2}, y_{n+3}\}$ is a positive semicycle. Now, according to the assumption when $k = 4 \times 1 - 1 = 3$, we have $\frac{y_{n+3}}{y_{n+2}} < 1$ and as a result we get

$$y_{n+4} = \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}\left(\frac{y_{n+3}}{y_{n+2}}\right)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} < \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}.(1)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = 1.$$

Then $y_{n+4} < 1$. It is obvious that $y_{n+4} < y_{n+3}$, which gives that $\frac{y_{n+4}}{y_{n+3}} < 1$, then

$$y_{n+5} = \frac{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}\left(\frac{y_{n+4}}{y_{n+3}}\right)}{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}} < \frac{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}.(1)}{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}} = 1.$$

Then $y_{n+5} < 1$, $\{y_{n+4}, y_{n+5}\}$ is a negative semicycle.

$$y_{n+6} = \frac{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}\left(\frac{y_{n+5}}{y_{n+4}}\right)}{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}} > \frac{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}.(1)}{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}} = 1.$$

Since for $k = 4 \times 1 + 1 = 5$ we have that $\frac{y_{n+5}}{y_{n+4}} > 1$ as in the assumption, then $y_{n+6} < 1$. And

$$y_{n+7} = \frac{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}} \left(\frac{y_{n+6}}{y_{n+5}} \right)}{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}} > \frac{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}.(1)}{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}} = 1.$$

Since $y_{n+6} > y_{n+5}$ we have $\frac{y_{n+6}}{y_{n+5}} > 1$, then $y_{n+7} > 1$. Thus, $\{y_{n+6}, y_{n+7}\}$ is a positive semicycle.
When $k = 4 \times 2 - 1 = 7$, then $\frac{y_{n+7}}{y_{n+6}} < 1$ and we get

$$y_{n+8} = \frac{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}} \left(\frac{y_{n+7}}{y_{n+6}} \right)}{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}} < \frac{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}.(1)}{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}} = 1.$$

Now, $y_{n+8} < 1$ and $y_{n+7} > 1$, then $\frac{y_{n+8}}{y_{n+7}} < 1$, as a result

$$y_{n+9} = \frac{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+7}} \left(\frac{y_{n+8}}{y_{n+7}} \right)}{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+8}}} < \frac{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+7}}.(1)}{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+7}}} = 1.$$

As a result, $\{y_{n+8}, y_{n+9}\}$ is a negative semicycle. If $k = 4 \times 2 + 1 = 9$, then $\frac{y_{n+9}}{y_{n+8}} > 1$ and

$$y_{n+10} = \frac{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}} \left(\frac{y_{n+9}}{y_{n+8}} \right)}{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}} > \frac{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}.(1)}{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}} = 1.$$

Clearly, $y_{n+11} > 1$. The semicycle $\{y_{n+10}, y_{n+11}\}$ is a positive semicycle.
The result says $\{y_{n+4l}, y_{n+4l+1}\}$ is a negative semicycle, $l = 1, 2, 3, ...,$ and $\{y_{n+4l+2}, y_{n+4l+3}\}$ is a positive semicycle, $l = 0, 1, 2, ...$
Now, setting $l = 0$ in the second semicycle gives the positive semicycle $\{y_{n+2}, y_{n+3}\}$ if we set $l = 1$ in the first semicycle, we get the negative semicycle $\{y_{n+4}, y_{n+5}\}$. If $l = 1$ in the second semicycle, we get the positive semicycle $\{y_{n+6}, y_{n+7}\}$, taking $l = 2$ in the first semicycle gives the negative semicycle $\{y_{n+8}, y_{n+9}\}$. $l = 2$ in the second semicycle produces the positive semicycle $\{y_{n+10}, y_{n+11}\}$, and these results match with the previous conclusions.
Assume that the result holds for $l - 1$. We prove it for l.
If for $k = 4l - 1$, $l = 1, 2, 3, ...,$ $\frac{y_{n+k}}{y_{n+k-1}} < 1$, and if for $k = 4l + 1$, $l = 0, 1, 2, ...,$

$\frac{y_{n+k}}{y_{n+k-1}} > 1$, $\{y_{n+4l}, y_{n+4l+1}\}$ is a negative semicycle and $\{y_{n+4l+2}, y_{n+4l+3}\}$ is a positive semicycle. In other words,

$$y_{n+k+1} = \frac{p_{n+k} + \frac{\bar{x}_{n+k}}{\bar{x}_{n+k-1}}\left(\frac{y_{n+k}}{y_{n+k-1}}\right)}{p_{n+k} + \frac{\bar{x}_{n+k}}{\bar{x}_{n+k-1}}}.$$

For $k = 4l - 1$

$$y_{n+4l} = \frac{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}\left(\frac{y_{n+4l-1}}{y_{n+4l-2}}\right)}{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}} < \frac{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}.(1)}{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}} = 1.$$

Consequently, $y_{n+4l+1} < 1$ by induction hypothesis. Now, using the second assumption we have

$$y_{n+4l+2} = \frac{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}\left(\frac{y_{n+4l+1}}{y_{n+4l}}\right)}{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}} > \frac{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}.(1)}{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}} = 1.$$

It is clear that $y_{n+4l+3} > 1$. As a result, $\{y_{n+4l}, y_{n+4l+1}\}$ is a negative semicycle and $\{y_{n+4l+2}, y_{n+4l+3}\}$ is a positive semicycle. ■

Theorem 4.2.5 *Let $\{y_n\}$ be a solution of Eq.(4.1). If there exists n such that $y_n < y_{n+1} < 1$ and $\frac{y_{n+k}}{y_{n+k-1}} < 1$ for $k = 4l - 1$, where $l = 1, 2, 3, ...$, and $\frac{y_{n+k}}{y_{n+k-1}} > 1$ for $k = 4l + 1$, where $l = 0, 1, 2, ...$, then $\{y_n\}$ is an oscillatory solution in which $\{y_{n+4l}, y_{n+4l+1}\}$ where $l = 1, 2, 3, ...$ gives all negative semicycles except the first and $\{y_{n+4l+2}, y_{n+4l+3}\}$ where $l = 0, 1, 2, ...$ gives all positive semicycles.*

Proof. We proceed by induction. $y_n < y_{n+1}$, then $\frac{y_{n+1}}{y_n} > 1$, which gives

$$y_{n+2} = \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}\left(\frac{y_{n+1}}{y_n}\right)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} > \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}.(1)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} = 1.$$

Then $y_{n+2} > 1$.
It is clear that $y_{n+2} > y_{n+1}$, then $\frac{y_{n+2}}{y_{n+1}} > 1$, this implies that

$$y_{n+3} = \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}\left(\frac{y_{n+2}}{y_{n+1}}\right)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} > \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}.(1)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = 1.$$

Then $y_{n+3} > 1$, and $\{y_{n+2}, y_{n+3}\}$ is a positive semicycle. Now, when $k = 4 \times 1 - 1 = 3$, according to the assumption $\frac{y_{n+3}}{y_{n+2}} < 1$, then

$$y_{n+4} = \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}\left(\frac{y_{n+3}}{y_{n+2}}\right)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} < \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}.(1)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = 1.$$

Then $y_{n+4} < 1$. It is obvious that $y_{n+4} < y_{n+3}$, which gives that $\frac{y_{n+4}}{y_{n+3}} < 1$, then

$$y_{n+5} = \frac{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}\left(\frac{y_{n+4}}{y_{n+3}}\right)}{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}} < \frac{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}.(1)}{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}} = 1.$$

Then $y_{n+5} < 1$ and $\{y_{n+4}, y_{n+5}\}$ is a negative semicycle.
When $k = 4 \times 1 + 1 = 5$, $\frac{y_{n+5}}{y_{n+4}} > 1$, then

$$y_{n+6} = \frac{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}\left(\frac{y_{n+5}}{y_{n+4}}\right)}{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}} > \frac{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}.(1)}{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}} = 1.$$

$$y_{n+7} = \frac{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}\left(\frac{y_{n+6}}{y_{n+5}}\right)}{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}} > \frac{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}.(1)}{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}} = 1.$$

$y_{n+6} > y_{n+5}$, consequently $\frac{y_{n+6}}{y_{n+5}} > 1$ and $y_{n+7} > 1$. As a result $\{y_{n+6}, y_{n+7}\}$ is a positive semicycle.
When $k = 4 \times 2 - 1 = 7$, then $\frac{y_{n+7}}{y_{n+6}} < 1$ and we get

$$y_{n+8} = \frac{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}\left(\frac{y_{n+7}}{y_{n+6}}\right)}{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}} < \frac{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}.(1)}{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}} = 1.$$

Now, $y_{n+8} < 1$ and $y_{n+7} > 1$, then $\frac{y_{n+8}}{y_{n+7}} < 1$, as a result

$$y_{n+9} = \frac{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+7}}\left(\frac{y_{n+8}}{y_{n+7}}\right)}{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+8}}} < \frac{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+7}}.(1)}{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+7}}} = 1.$$

As a result, $\{y_{n+8}, y_{n+9}\}$ is a negative semicycle. If $k = 4 \times 2 + 1 = 9$, then $\frac{y_{n+9}}{y_{n+8}} > 1$, then

$$y_{n+10} = \frac{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}\left(\frac{y_{n+9}}{y_{n+8}}\right)}{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}} > \frac{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}.(1)}{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}} = 1.$$

Clearly, $y_{n+11} > 1$. The semicycle $\{y_{n+10}, y_{n+11}\}$ is a positive semicycle.

Then $\{y_n, y_{n+1}\}$ is a negative semicycle and this results from the assumption $y_n < y_{n+1} < 1$, $\{y_{n+2}, y_{n+3}\}$ is a positive semicycle, $\{y_{n+4}, y_{n+5}\}$ is a negative semicycle, $\{y_{n+6}, y_{n+7}\}$ is positive semicycle, $\{y_{n+8}, y_{n+9}\}$ is a negative semicycle, and $\{y_{n+10}, y_{n+11}\}$ is a positive semicycle. If you used $l = 0$ you will get the second semicycle, and $l = 1$ gives the second two semicycles, and $l = 2$ gives the third two semicycles.
Assume that the result holds for $l - 1$. We prove it for l.
If for $k = 4l - 1$, $l = 1, 2, 3, ...$, $\frac{y_{n+k}}{y_{n+k-1}} < 1$, and if for $k = 4l + 1$, $l = 0, 1, 2, ...$, $\frac{y_{n+k}}{y_{n+k-1}} > 1$, $\{y_{n+4l}, y_{n+4l+1}\}$ is a negative semicycle and $\{y_{n+4l+2}, y_{n+4l+3}\}$ is a positive semicycle. In other words,

$$y_{n+k+1} = \frac{p_{n+k} + \frac{\bar{x}_{n+k}}{\bar{x}_{n+k-1}}\left(\frac{y_{n+k}}{y_{n+k-1}}\right)}{p_{n+k} + \frac{\bar{x}_{n+k}}{\bar{x}_{n+k-1}}}.$$

For $k = 4l - 1$

$$y_{n+4l} = \frac{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}\left(\frac{y_{n+4l-1}}{y_{n+4l-2}}\right)}{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}} < \frac{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}.(1)}{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}} = 1.$$

consequently, $y_{n+4l+1} < 1$ by induction hypothesis. Now, using the second assumption we have

$$y_{n+4l+2} = \frac{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}\left(\frac{y_{n+4l+1}}{y_{n+4l}}\right)}{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}} > \frac{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}.(1)}{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}} = 1.$$

It is clear that $y_{n+4l+3} > 1$. As a result, $\{y_{n+4l}, y_{n+4l+1}\}$ is a negative semicycle and $\{y_{n+4l+2}, y_{n+4l+3}\}$ is a positive semicycle. ■

Theorem 4.2.6 *Let $\{y_n\}$ be a solution of Eq.(4.1). If there exists n such that $y_{n+1} < y_n < 1$ and $\frac{y_{n+k}}{y_{n+k-1}} < 1$ for $k = 2$ and $k = 4l + 1$, where $l = 0, 1, 2, ...$, and $\frac{y_{n+k}}{y_{n+k-1}} > 1$ for $k = 4l - 1$, where $l = 1, 2, 3, ...$, then $\{y_{n+4l}, y_{n+4l+1}\}$ where $l = 1, 2, 3, ...$ gives all positive semicycles, and $\{y_{n+4l+2}, y_{n+4l+3}\}$ where $l = 0, 1, 2, ...$ gives all negative semicycles except perhaps the first.*

Proof. We proceed by induction. $y_{n+1} < y_n$, then $\frac{y_{n+1}}{y_n} < 1$, which gives

$$y_{n+2} = \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}\left(\frac{y_{n+1}}{y_n}\right)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} < \frac{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}.(1)}{p_{n+1} + \frac{\bar{x}_{n+1}}{\bar{x}_n}} = 1.$$

Then $y_{n+2} < 1$.
It is assumed that $\frac{y_{n+k}}{y_{n+k-1}} < 1$ for $k = 2$, then $\frac{y_{n+2}}{y_{n+1}} < 1$, this implies that

$$y_{n+3} = \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}\left(\frac{y_{n+2}}{y_{n+1}}\right)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} < \frac{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}.(1)}{p_{n+2} + \frac{\bar{x}_{n+2}}{\bar{x}_{n+1}}} = 1.$$

Then $y_{n+3} < 1$, as a result $\{y_{n+2}, y_{n+3}\}$ is a negative semicycle. Now, when $k = 4 \times 1 - 1 = 3$, according to the assumption $\frac{y_{n+3}}{y_{n+2}} > 1$

$$y_{n+4} = \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}\left(\frac{y_{n+3}}{y_{n+2}}\right)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} > \frac{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}.(1)}{p_{n+3} + \frac{\bar{x}_{n+3}}{\bar{x}_{n+2}}} = 1.$$

Then $y_{n+4} > 1$. It is obvious that $y_{n+4} > y_{n+3}$, which gives that $\frac{y_{n+4}}{y_{n+3}} > 1$, then

$$y_{n+5} = \frac{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}\left(\frac{y_{n+4}}{y_{n+3}}\right)}{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}} > \frac{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}.(1)}{p_{n+4} + \frac{\bar{x}_{n+4}}{\bar{x}_{n+3}}} = 1.$$

Then $y_{n+5} > 1$, $\{y_{n+4}, y_{n+5}\}$ is a positive semicycle.
When $k = 4 \times 1 + 1 = 5$, $\frac{y_{n+5}}{y_{n+4}} < 1$, then

$$y_{n+6} = \frac{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}\left(\frac{y_{n+5}}{y_{n+4}}\right)}{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}} < \frac{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}.(1)}{p_{n+5} + \frac{\bar{x}_{n+5}}{\bar{x}_{n+4}}} = 1.$$

$$y_{n+7} = \frac{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}\left(\frac{y_{n+6}}{y_{n+5}}\right)}{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}} < \frac{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}.(1)}{p_{n+6} + \frac{\bar{x}_{n+6}}{\bar{x}_{n+5}}} = 1.$$

Since $y_{n+6} < y_{n+5}$, so $\frac{y_{n+6}}{y_{n+5}} < 1$, then $y_{n+7} < 1$ and $\{y_{n+6}, y_{n+7}\}$ is a negative semicycle.
When $k = 4 \times 2 - 1 = 7$, then $\frac{y_{n+7}}{y_{n+6}} > 1$ and we get

$$y_{n+8} = \frac{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}\left(\frac{y_{n+7}}{y_{n+6}}\right)}{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}} > \frac{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}.(1)}{p_{n+7} + \frac{\bar{x}_{n+7}}{\bar{x}_{n+6}}} = 1.$$

Now, $y_{n+8} > 1$ and $y_{n+7} < 1$, then $\frac{y_{n+8}}{y_{n+7}} > 1$, as a result

$$y_{n+9} = \frac{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+7}}\left(\frac{y_{n+8}}{y_{n+7}}\right)}{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+8}}} > \frac{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+7}}.(1)}{p_{n+8} + \frac{\bar{x}_{n+8}}{\bar{x}_{n+7}}} = 1.$$

As a result, $\{y_{n+8}, y_{n+9}\}$ is a positie semicycle. If $k = 4 \times 2 + 1 = 9$, then $\frac{y_{n+9}}{y_{n+8}} < 1$, then

$$y_{n+10} = \frac{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}\left(\frac{y_{n+9}}{y_{n+8}}\right)}{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}} < \frac{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}.(1)}{p_{n+9} + \frac{\bar{x}_{n+9}}{\bar{x}_{n+8}}} = 1.$$

Clearly, $y_{n+11} < 1$. The semicycle $\{y_{n+10}, y_{n+11}\}$ is a negative semicycle.
Now, we have two semicycles $\{y_{n+4l}, y_{n+4l+1}\}$, and $\{y_{n+4l+2}, y_{n+4l+3}\}$, $l = 0$ in the second semicycle gives the negative semicycle $\{y_{n+2}, y_{n+3}\}$, $l = 1$ in the first one gives the positive semicycle $\{y_{n+4}, y_{n+5}\}$ and in the second one gives the negative semicycle $\{y_{n+6}, y_{n+7}\}$, $l = 2$ in the first one gives the positive semicycle $\{y_{n+8}, y_{n+9}\}$ and in the second one gives the negative semicycle $\{y_{n+10}, y_{n+11}\}$
Assume that the result holds for $l - 1$. We prove it for l.
If for $k = 4l - 1$, $l = 1, 2, 3, ...,$ $\frac{y_{n+k}}{y_{n+k-1}} > 1$, and if for $k = 4l + 1$, $l = 0, 1, 2, ...$, $\frac{y_{n+k}}{y_{n+k-1}} < 1$, then $\{y_{n+4l}, y_{n+4l+1}\}$ gives all positive semicycles, and $\{y_{n+4l+2}, y_{n+4l+3}\}$ gives all negative semicycles except perhaps the first. In other words,

$$y_{n+k+1} = \frac{p_{n+k} + \frac{\bar{x}_{n+k}}{\bar{x}_{n+k-1}}\left(\frac{y_{n+k}}{y_{n+k-1}}\right)}{p_{n+k} + \frac{\bar{x}_{n+k}}{\bar{x}_{n+k-1}}}.$$

For $k = 4l - 1$

$$y_{n+4l} = \frac{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}\left(\frac{y_{n+4l-1}}{y_{n+4l-2}}\right)}{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}} > \frac{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}.(1)}{p_{n+4l-1} + \frac{\bar{x}_{n+4l-1}}{\bar{x}_{n+4l-2}}} = 1.$$

consequently, $y_{n+4l+1} > 1$ by induction hypothesis. Now, using the second assumption we have

$$y_{n+4l+2} = \frac{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}\left(\frac{y_{n+4l+1}}{y_{n+4l}}\right)}{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}} < \frac{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}.(1)}{p_{n+4l+1} + \frac{\bar{x}_{n+4l+1}}{\bar{x}_{n+4l}}} = 1.$$

It is clear that $y_{n+4l+3} < 1$. As a result, $\{y_{n+4l}, y_{n+4l+1}\}$ is a positive semicycle and $\{y_{n+4l+2}, y_{n+4l+3}\}$ is a negative semicycle. ∎

4.3 Applications

Definition 4.3.1 *We say that $\{p_n\}$ is periodic with prime period k if*

$$p_{n+k} = p_n \text{ for } n = -1, 0, ...$$

Assume that $\{p_n\}$ is periodic with period k.

$$p = \liminf_{n\to\infty} p_n,$$

and

$$q = \limsup_{n\to\infty} p_n.$$

Lemma 4.3.1 *A necessary condition for the existence of a periodic solution $\{x_n\}$ of Eq.(4.1) with prime period k is that $\{p_n\}$ is periodic with period k.*

Proof. Assume that x_n is a periodic solution with prime period k, so we have $x_{n+k} = x_n$, for $n = -1, 0, ...$, we have

$$x_{n+k+1} = p_{n+k} + \frac{x_{n+k}}{x_{n+k-1}}.$$

So we get that

$$p_{n+k} = x_{n+k+1} - \frac{x_{n+k}}{x_{n+k-1}} = x_{n+1} - \frac{x_n}{x_{n-1}} = p_n.$$

Then $p_{n+k} = p_n$, this means that $\{p_n\}$ is periodic with prime period k. ■

Theorem 4.3.1 *Assume that p_n is periodic with prime period k, and let $1 < p < q$. Then there exists a positive periodic solution $\{\bar{x}_n\}$ of Eq.(4.1) with prime period k.*

Proof. We aim here to show that there is a periodic solution for Eq.(4.1) with period k. It is enough to show that the system has a positive solution.

$$x_1 = p_0 + \frac{x_0}{x_{-1}} = p_k + \frac{x_k}{x_{k-1}}.$$

$$x_2 = p_1 + \frac{x_1}{x_0} = p_1 + \frac{x_1}{x_k}.$$

$$x_3 = p_2 + \frac{x_2}{x_1}.$$

$$\vdots$$

$$x_k = p_{k-1} + \frac{x_{k-1}}{x_{k-2}}.$$

Define a function $F : R_+^k \to R_+^k$ such that,

$$F(u_1, u_2, ..., u_k) = \left(p_k + \frac{u_k}{u_{k-1}}, p_1 + \frac{u_1}{u_k}, ..., p_{k-1} + \frac{u_{k-1}}{u_{k-2}}\right).$$

In addition define an interval $I = \left[\frac{pq-1}{q-1}, \frac{pq-1}{p-1}\right]$. Now, we aim to show that I^k is invariant under the function F. If $u_1, u_2, ..., u_k \in I$, we have

$$\begin{aligned} p_i + \frac{u_i}{u_j} \\ \leq q + \frac{\frac{pq-1}{p-1}}{\frac{pq-1}{q-1}} \\ = q + \frac{q-1}{p-1} \\ = \frac{qp - q + q - 1}{p-1} \\ = \frac{pq-1}{p-1}, \\ \text{for } \ i = 1, 2, ..., k, j = (i-1) \mod k \end{aligned}$$

since the above system is periodic of period k ,

$$\begin{aligned} p_i + \frac{u_i}{u_j} \\ \geq p + \frac{\frac{pq-1}{q-1}}{\frac{pq-1}{p-1}} \\ = p + \frac{p-1}{q-1} \\ = \frac{pq - p + p - 1}{q-1} \\ = \frac{pq-1}{q-1}, \\ \text{for } \ i = 1, 2, ..., k, j = (i-1) \mod k \end{aligned}$$

for the same reason as above.

Then $p_i + \frac{u_i}{u_j} \in I$ for $i = 1, ..., k, j = (i-1) \mod k$. So I^k is invariant under the function F. Now, we have $F : I^k \to I^k$ and F is continuous on I^k and I^k is convex and compact. Then, by Brower Fixed Point Theorem F has a fixed point in I^k.
Assume that the fixed point is $(\bar{u}_1, \bar{u}_2, ..., \bar{u}_k) \in I^k$. Define the sequence

$$\bar{x}_{-1} = \bar{u}_{k-1}, \ \ \bar{x}_0 = \bar{u}_k \ \text{ and } \ \bar{x}_{mk+i} = \bar{u}_i, \ \text{ for } \ i = 1, 2, ..., \ m = 0, 1, ...$$

This sequence satisfies the Eq.(4.1) and is periodic with period k.
■

Corollary 4.3.1 *Assume that $\{p_n\}$ is a convergent sequence and*

$$\lim_{n\to\infty} p_n = p > 1.$$

Then every solution $\{x_n\}$ of Eq.(4.1) is convergent and

$$\lim_{n\to\infty} x_n = p + 1.$$

Proof. $\{p_n\}$ is bounded so $\{x_n\}$ is bounded and persists according to (4.1.1). And we have

$$\lambda = \liminf_{n\to\infty} x_n \ \text{ and } \ \mu = \limsup_{n\to\infty} x_n.$$

And

$$p = \liminf_{n\to\infty} p_n \ \text{ and } \ q = \limsup_{n\to\infty} p_n.$$

And from Lemma (4.1.2) we have that

$$\frac{pq-1}{q-1} \leq \lambda \leq \mu \leq \frac{pq-1}{p-1}.$$

$\{p_n\}$ is convergent so $p = \liminf_{n\to\infty} p_n = \limsup_{n\to\infty} p_n = q$. Then we have that

$$p + 1 = \frac{p^2-1}{p-1} \leq \lambda \leq \mu \leq \frac{p^2-1}{p-1} = p + 1.$$

So we have $\lambda = \mu = p + 1$. Then as a result we get $\lim_{n\to\infty} x_n = p + 1$. ■

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